# Overlapping Edge Unfoldings for Archimedean Solids and (Anti)prisms

WALCOM2023

◎ Takumi SHIOTA and Toshiki SAITOH

Kyushu Institute of Technology, Japan

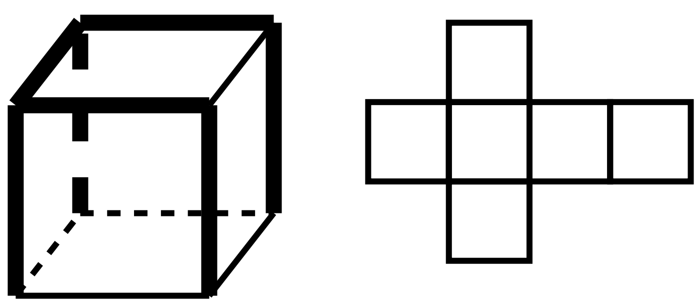
March 22, 2023

# Edge unfoldings

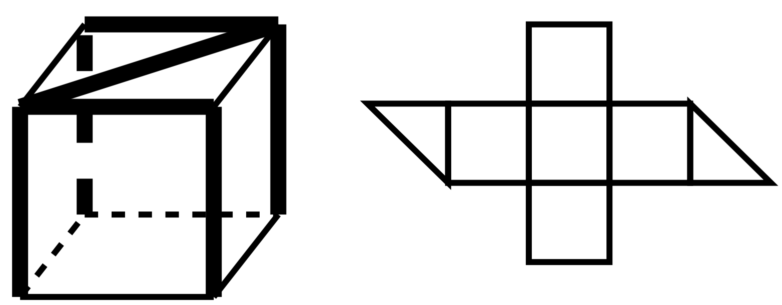
**Definition 1 [R. Uehara, 2018]**

An edge unfolding of the polyhedron is a flat polygon formed by cutting its edges and unfolding it into a plane.

Cutting along the thick line of each left cube …



(a) Edge unfolding



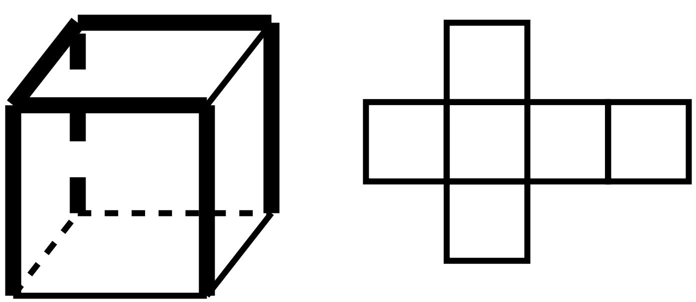
(b) Not edge unfolding

# Edge unfoldings

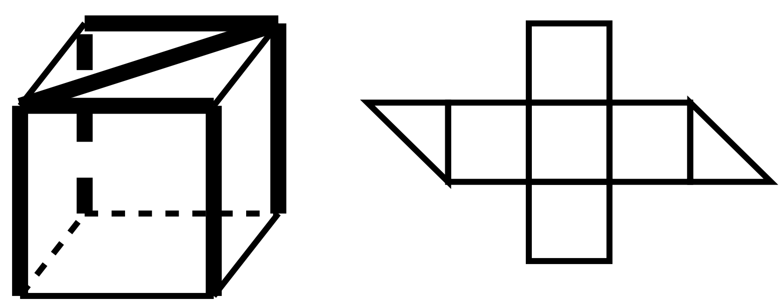
Definition 1 [R. Uehara, 2018]

An edge unfolding of the polyhedron is a flat polygon formed by cutting its edges and unfolding it into a plane.

Cutting along the thick line of each left cube …



(a) Edge unfolding



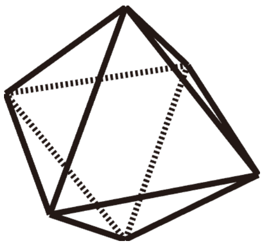
(b) Not edge unfolding

# Convex regular-faced polyhedra

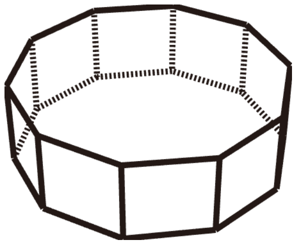
**Definition 2**

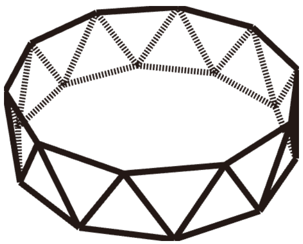
Convex regular-faced polyhedra are convex polyhedra with all faces are regular polygons.

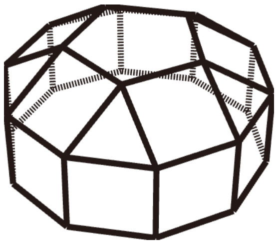
Categorized into 5 classes


Platonic solid


Archimedean solid

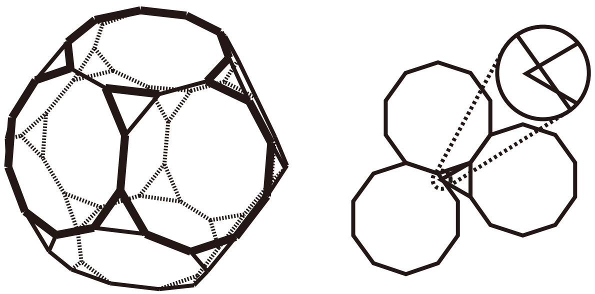

Archimedean prism


Archimedean antiprism

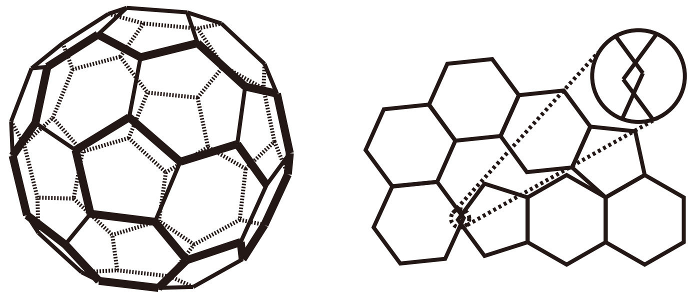

Johnson solid

# Overlapping edge unfoldings

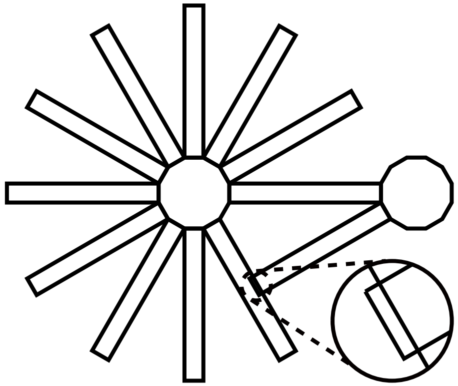
Overlapping edge unfoldings exist in some convex polyhedra.



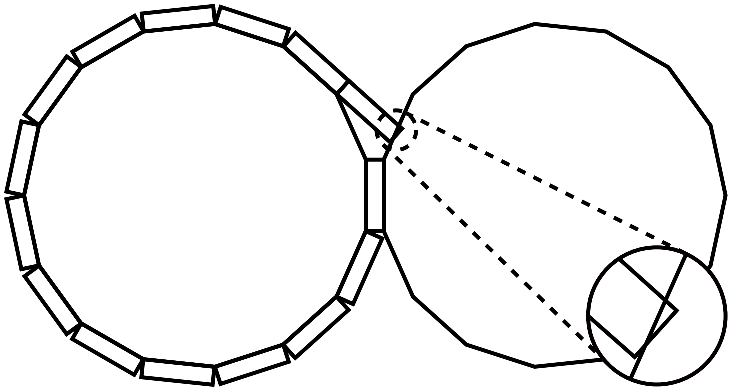
Truncated dodecahedron
[T. Horiyama and W. Shoji, 2011]



Truncated icosahedron
[T. Horiyama and W. Shoji, 2011]



12-gonal prism
[Schlickenrieder, 1997]



15-gonal prism
[Schlickenrieder, 1997]

# Background and our results

Investigate: convex regular-faced polyhedra

| Convex regular-faced polyhedra | Is there an overlapping edge unfolding? |
|---|---|
| Platonic solids (Total 5 types) | No [T. Horiyama and W. Shoji, 2011] |
| Archimedean solids (Total 13 types) | Yes (5 types) [T. Horiyama and W. Shoji, 2011]<br>No (5 types) [Hirose, 2015]<br>**Open Problem** (3 types) |
| $n$-gonal Archimedean prisms ( $n \geq 3$ ) | **Open Problem** |
| $n$-gonal Archimedean antiprisms ( $n \geq 3$ ) | **Open Problem** |
| Johnson solids (Total 92 types) | **Open Problem** |

# Background and our results

Investigate: convex regular-faced polyhedra

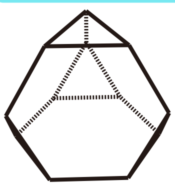
| **Convex regular-faced polyhedra** | **Is there an overlapping edge unfolding?** |
|---|---|
| Platonic solids<br>(Total 5 types) | No [T. Horiyama and W. Shoji, 2011] |
| Archimedean solids<br>(Total 13 types) | Yes (5 types) [T. Horiyama and W. Shoji, 2011]<br>No (5 types) [Hirose, 2015]<br>**No (2 types), Yes (1 type)** |
| $n$-gonal Archimedean prisms ( $n \geq 3$ ) | **No** $(3 \leq n \leq 23)$<br>**Yes** $(n \geq 24)$ |
| $n$-gonal Archimedean antiprisms ( $n \geq 3$ ) | **No** $(3 \leq n \leq 11)$<br>**Yes** $(n \geq 12)$ |
| Johnson solids<br>(Total 92 types) | **No (48 types)**<br>**Yes (44 types)** |

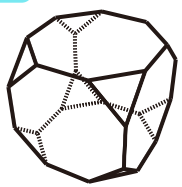
# Our results in Archimedean solids

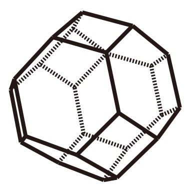
**Theorem 1**

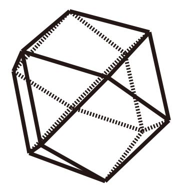
1. An icosidodecahedron and a rhombitruncated cuboctahedron have no overlapping edge unfoldings.
2. A snub cube has overlapping edge unfoldings.

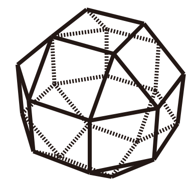
**No exist**


Truncated tetrahedron

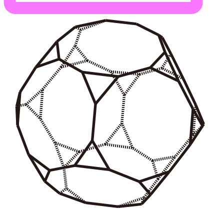

Truncated hexahedron

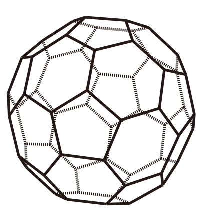

Truncated octahedron

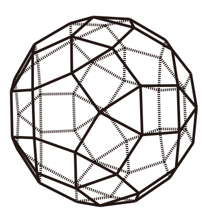

Cuboctahedron

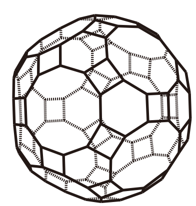

Rhombi cuboctahedron

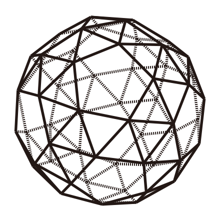
**Exist**


Truncated dodecahedron


Truncated icosahedron

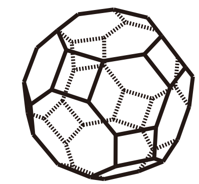

Rhombicosidode cahedron

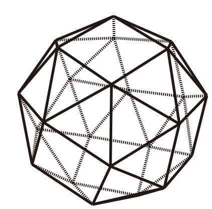

Rhombitruncated icosidodecahedron


Snub dodecahedron

**Open**


Icosidodecahedron


Rhombitruncated cuboctahedron


Snub cube

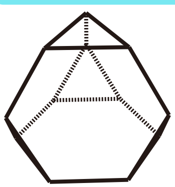
# Our results in Archimedean solids

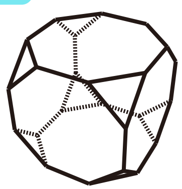
**Theorem 1**

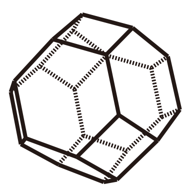
1. An icosidodecahedron and a rhombitruncated cuboctahedron have no overlapping edge unfoldings.
2. A snub cube has overlapping edge unfoldings.

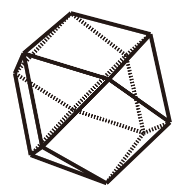
**No exist**

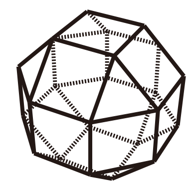
**New results**


Truncated tetrahedron

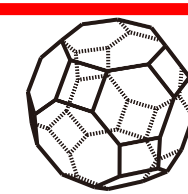

Truncated hexahedron


Truncated octahedron

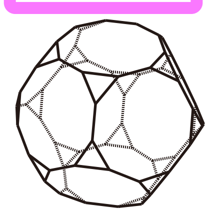

Cuboctahedron

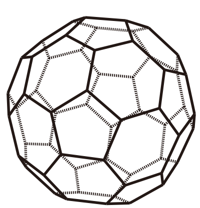

Rhombi cuboctahedron

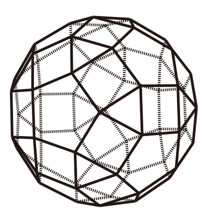

Icosidodecahedron

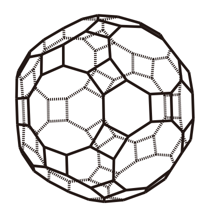

Rhombitruncated cuboctahedron

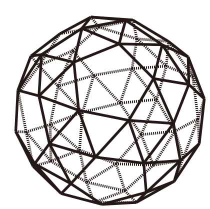
**Exist**

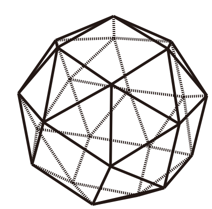

Truncated dodecahedron


Truncated icosahedron


Rhombicosidode cahedron


Rhombitruncated icosidodecahedron

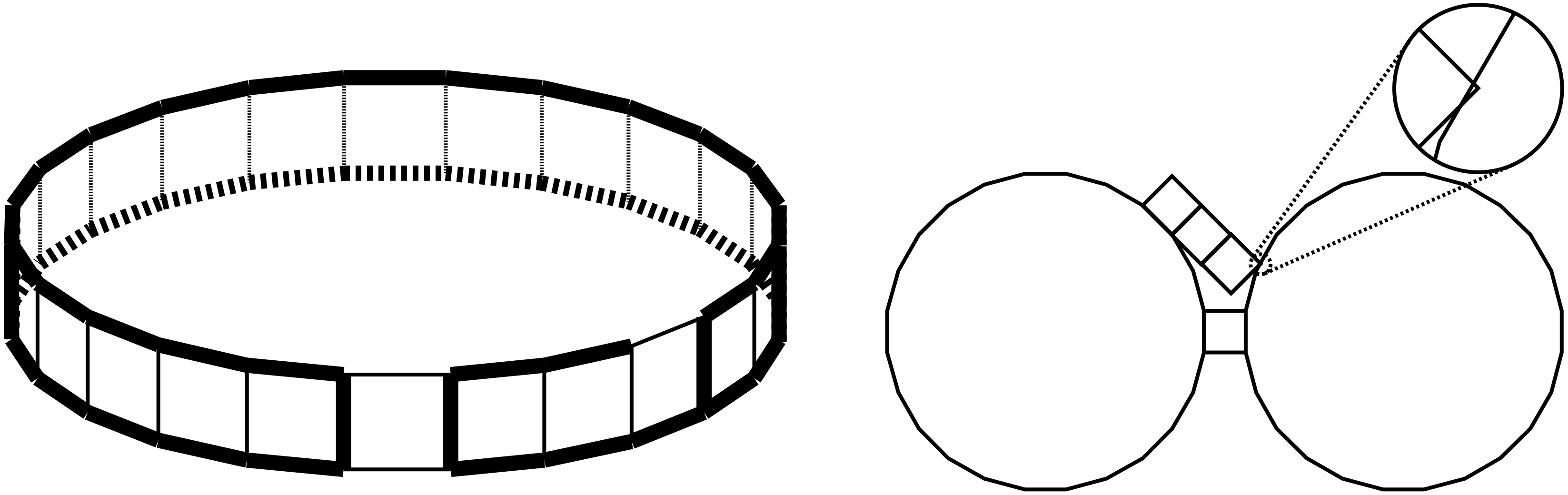

Snub dodecahedron


Snub cube

# Our results in Archimedean prisms

Theorem 2

1. Overlapping edge unfoldings do not exist for $n$-gonal Archimedean prisms for $3 \leq n \leq 23$.
2. Overlapping edge unfoldings exist in $n$-gonal Archimedean prisms for $n \geq 24$.

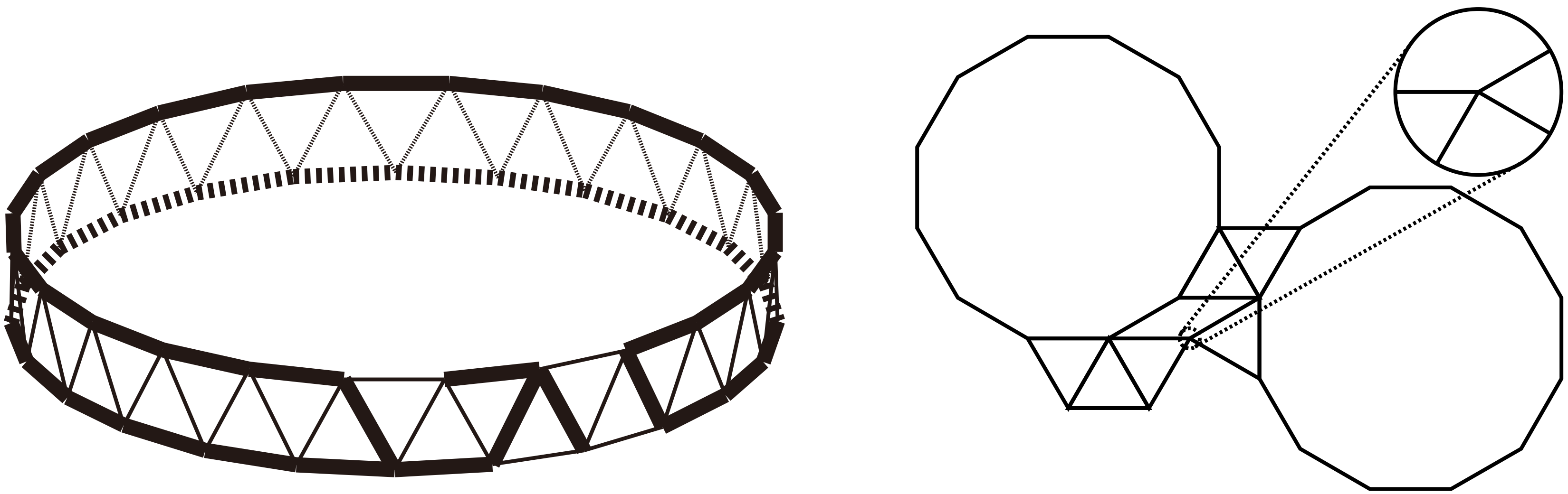


An overlapping edge unfolding in a $24$-gonal Archimedean prism

# Our results in Archimedean antiprisms

**Theorem 3**

1. Overlapping edge unfoldings do not exist for $n$-gonal Archimedean antiprisms for $3 \le n \le 11$.
2. Overlapping edge unfoldings exist in $n$-gonal Archimedean antiprisms for $n \ge 12$.



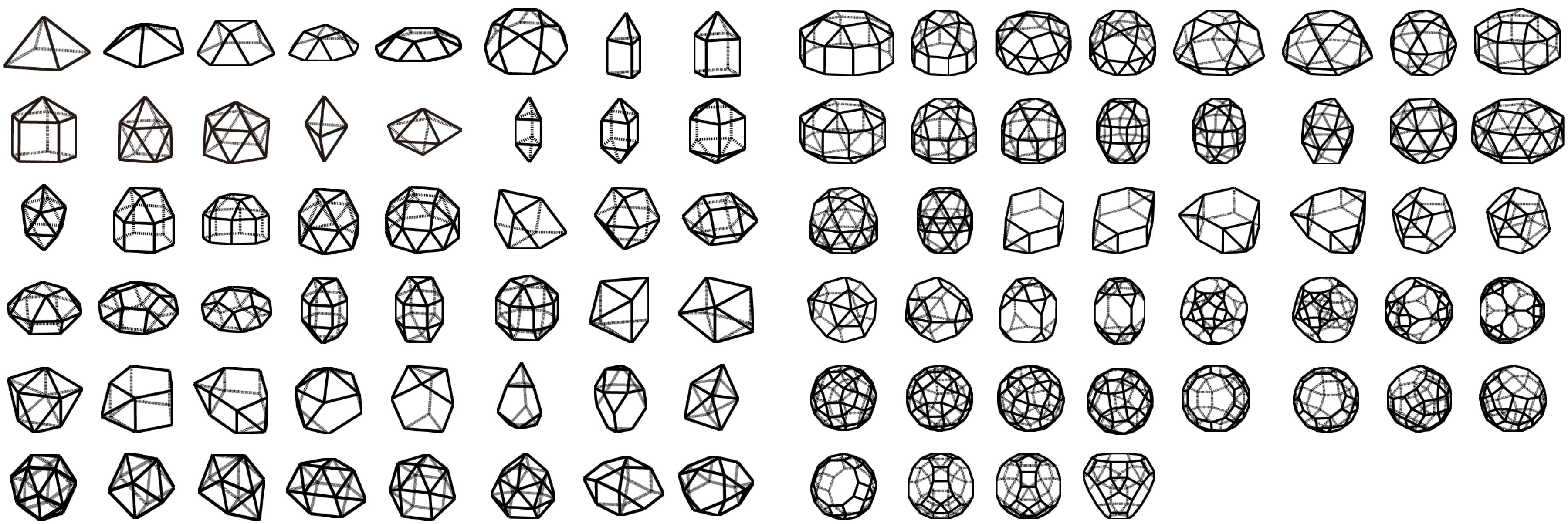
An overlapping edge unfolding in a $12$-gonal Archimedean antiprism

# Our Results in Johnson solids

Theorem 4

1. 48 Johnson solids do not have overlapping edge unfoldings.
2. 44 Johnson solids have overlapping edge unfoldings.

Open (92 types)



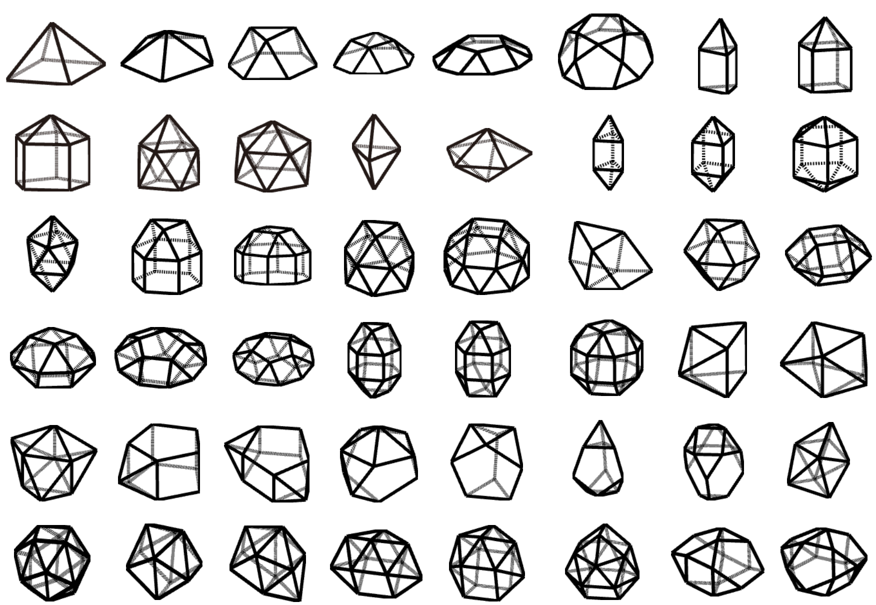
* Johnson Solid image files were used as published in https://mitani.cs.tsukuba.ac.jp/polyhedron/data/polyhedron.zip

# Our Results in Johnson solids

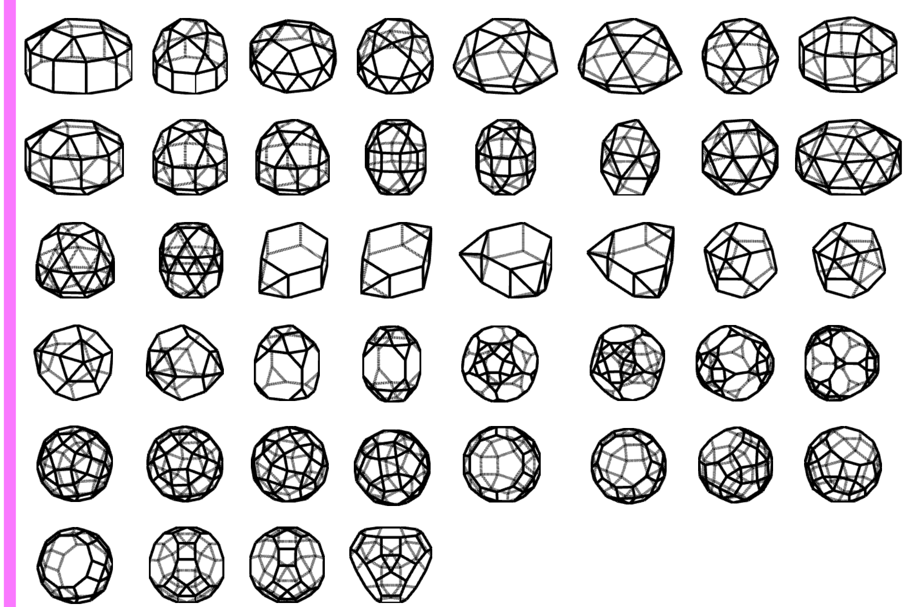
**Theorem 4**

1. 48 Johnson solids do not have overlapping edge unfoldings.
2. 44 Johnson solids have overlapping edge unfoldings.

No exist (48 types)



Exist (44 types)



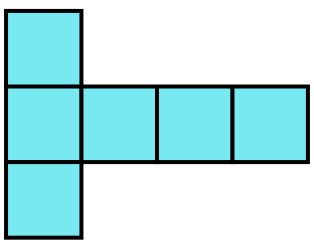
* Johnson Solid image files were used as published in https://mitani.cs.tsukuba.ac.jp/polyhedron/data/polyhedron.zip

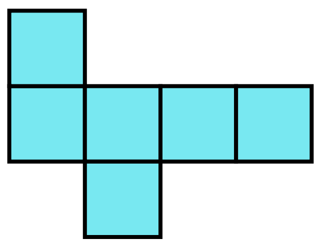
# Algorithm by T. Horiyama and W. Shoji

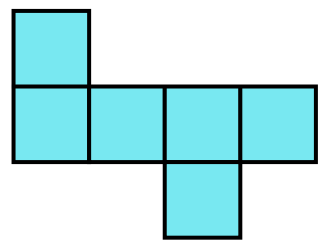
**Procedure [T. Horiyama and W. Shoji, 2011]**

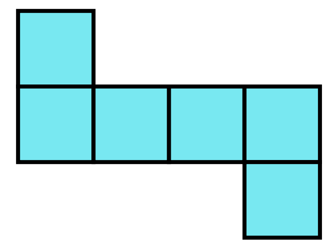
1. Enumerate the edge unfoldings of a polyhedron.
2. Check the overlapping for each unfolding.

(Ex.) Hexahedron (The number of edge unfoldings = 11)

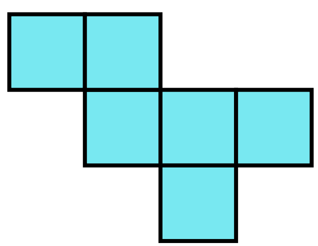

(1)

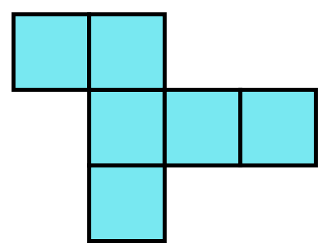

(2)

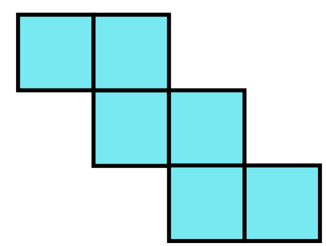

(3)


(4)

…


(9)


(10)

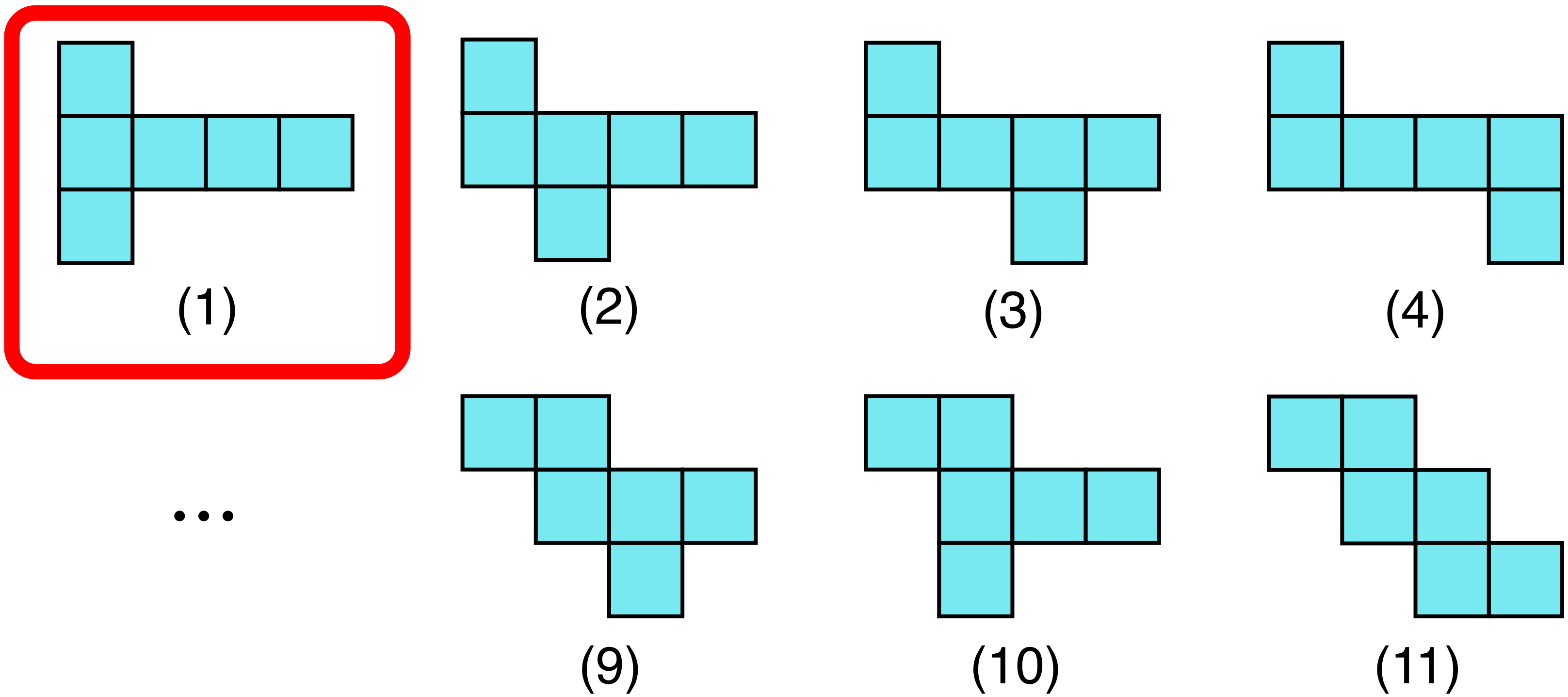

(11)

# Algorithm by T. Horiyama and W. Shoji

**Procedure [T. Horiyama and W. Shoji, 2011]**

1. Enumerate the edge unfoldings of a polyhedron.
2. Check the overlapping for each unfolding.

(Ex.) Hexahedron (The number of edge unfoldings = 11)

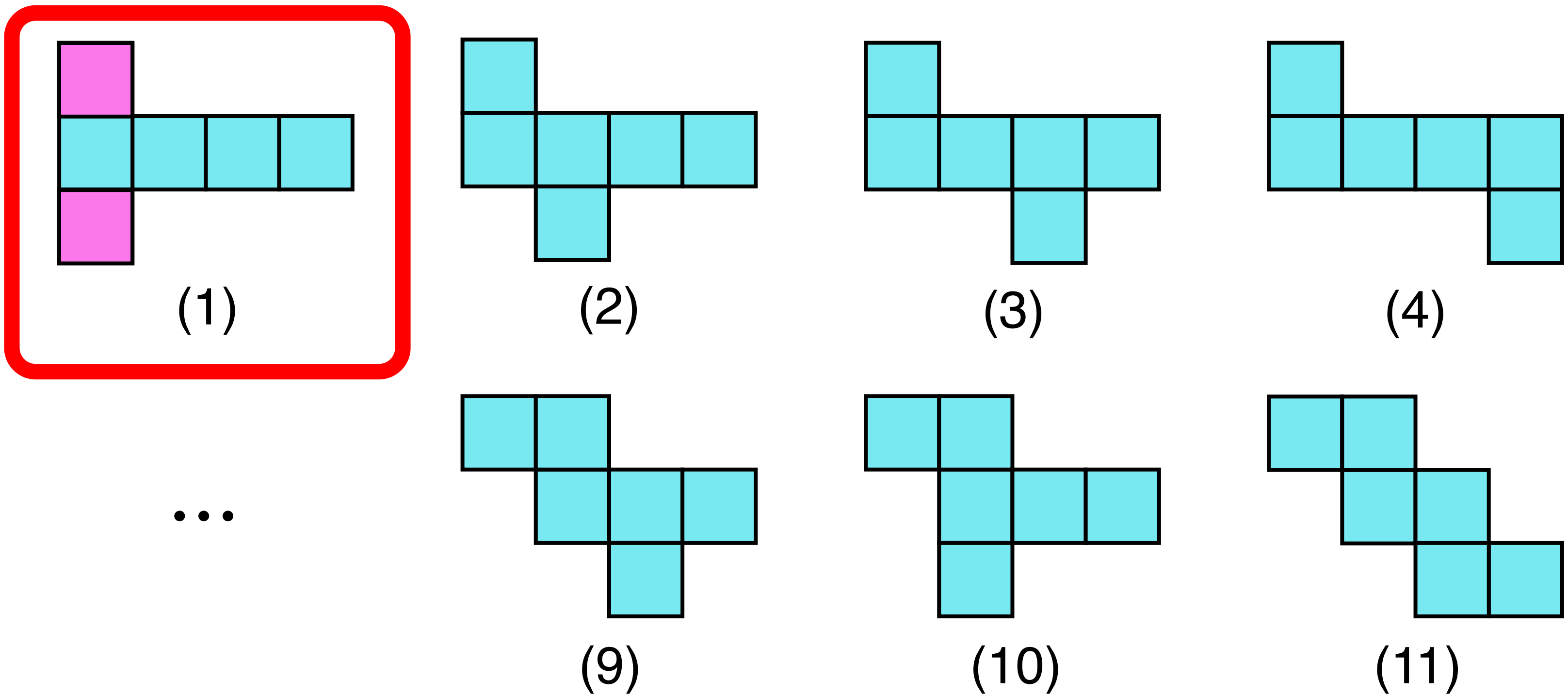


# Algorithm by T. Horiyama and W. Shoji

**Procedure [T. Horiyama and W. Shoji, 2011]**

1. Enumerate the edge unfoldings of a polyhedron.
2. Check the overlapping for each unfolding.

(Ex.) Hexahedron (The number of edge unfoldings = 11)

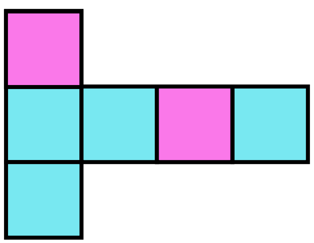


# Algorithm by T. Horiyama and W. Shoji

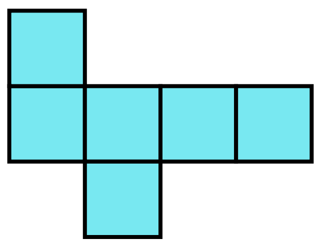
**Procedure [T. Horiyama and W. Shoji, 2011]**

1. Enumerate the edge unfoldings of a polyhedron.
2. Check the overlapping for each unfolding.

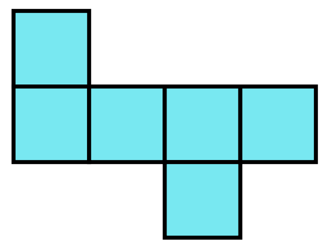
(Ex.) Hexahedron (The number of edge unfoldings = 11)



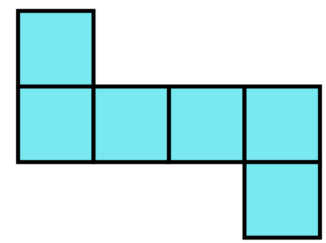
(1)



(2)

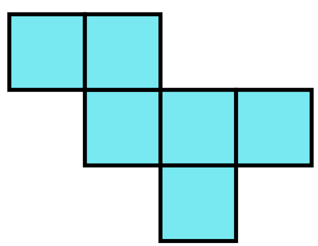


(3)

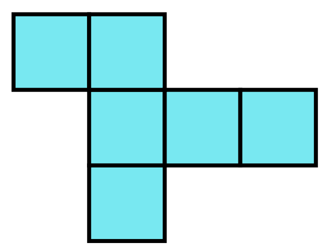


(4)

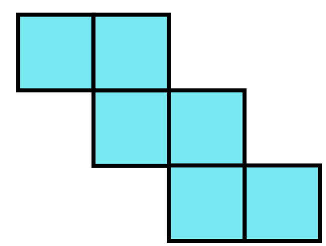
…



(9)



(10)



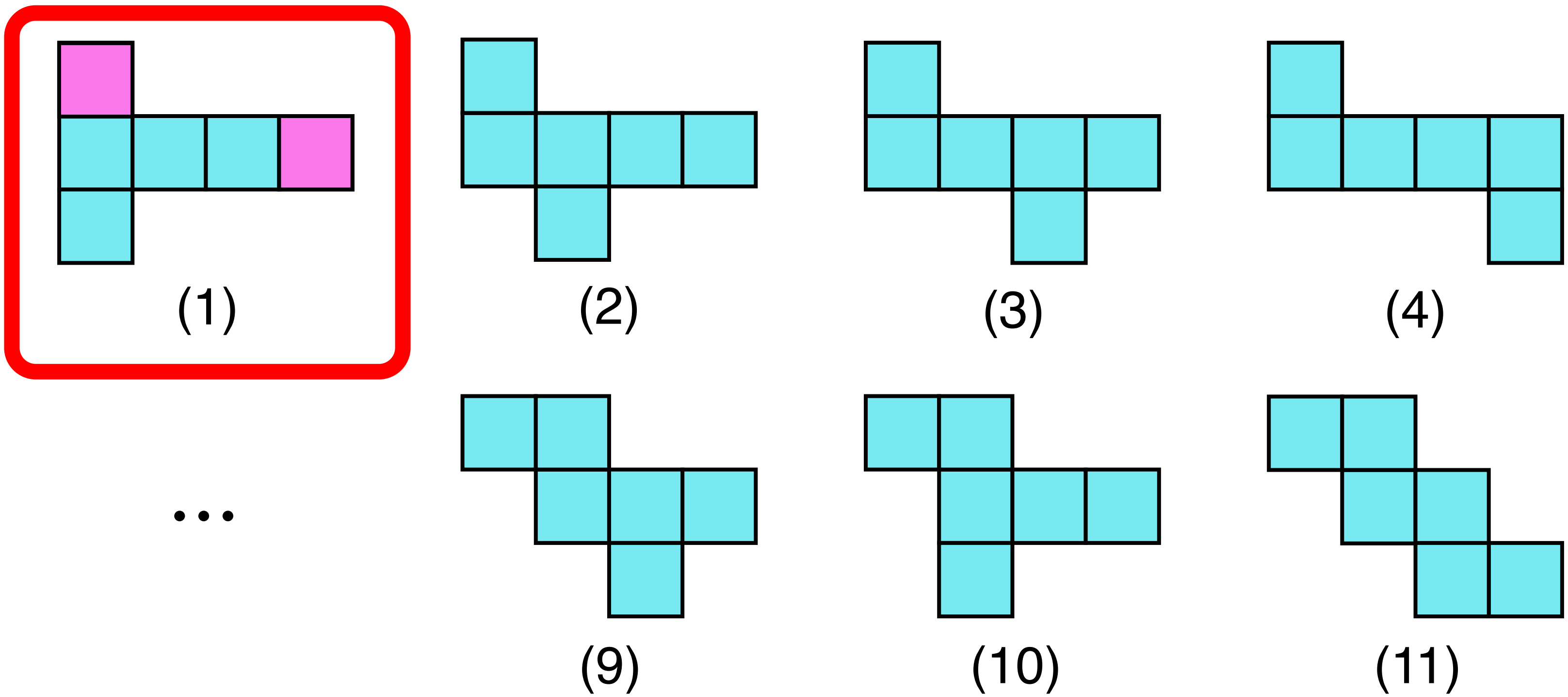
(11)

# Algorithm by T. Horiyama and W. Shoji

Procedure [T. Horiyama and W. Shoji, 2011]

1. Enumerate the edge unfoldings of a polyhedron.
2. Check the overlapping for each unfolding.

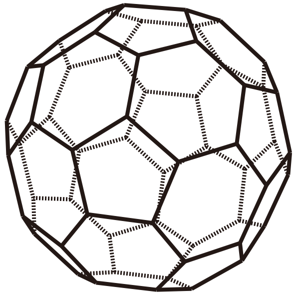
(Ex.) Hexahedron (The number of edge unfoldings = 11)



# Algorithm by T. Horiyama and W. Shoji

If we use [T. Horiyama and W. Shoji, 2011] algorithm…

①



Truncated icosahedron

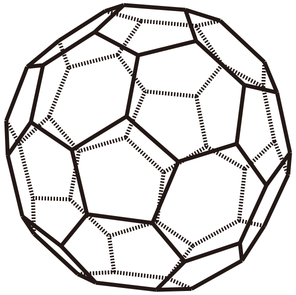
The number of edge unfoldings $\approx 3\times 10^{18}$

➡ 100 years to check!

# Algorithm by T. Horiyama and W. Shoji

If we use [T. Horiyama and W. Shoji, 2011] algorithm…

①



Truncated icosahedron

The number of edge unfoldings $\approx 3\times 10^{18}$

➡ 100 years to check!

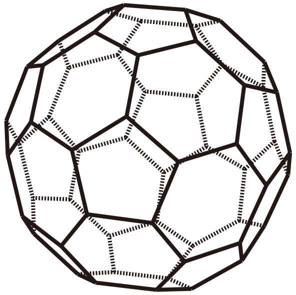
- The number of edge unfoldings is too huge.

  ➡ Cannot check the overlapping in a realistic time.

# Algorithm by T. Horiyama and W. Shoji

If we use [T. Horiyama and W. Shoji, 2011] algorithm…

①



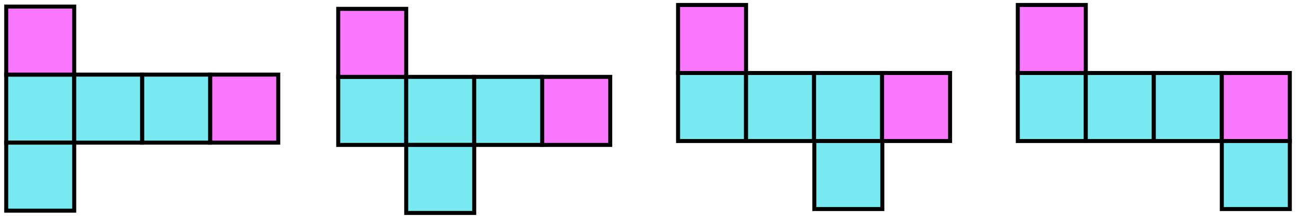
Truncated icosahedron

The number of edge unfoldings $\approx 3\times10^{18}$

➡ 100 years to check!

- The number of edge unfoldings is too huge.
  ➡ Cannot check the overlapping in a realistic time.

②

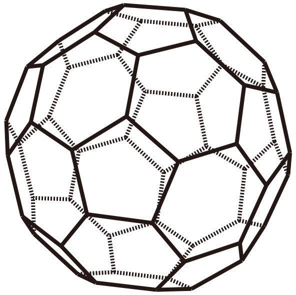


(1) (2) (3) (4)

# Algorithm by T. Horiyama and W. Shoji

If we use [T. Horiyama and W. Shoji, 2011] algorithm…

①



Truncated icosahedron

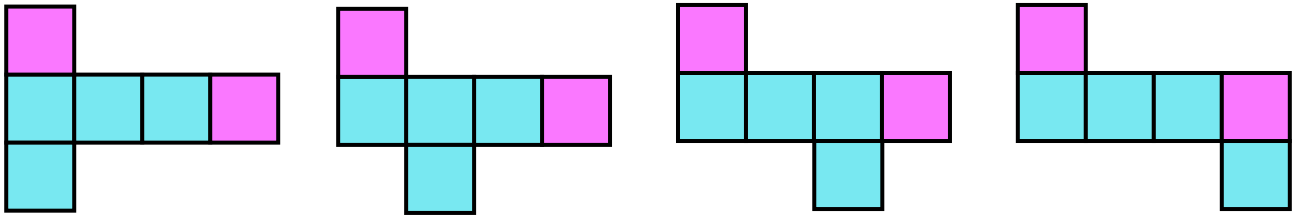
The number of edge unfoldings $\approx 3\times10^{18}$

➡ 100 years to check!

- The number of edge unfoldings is too huge.

  ➡ Cannot check the overlapping in a realistic time.

②



(1) (2) (3) (4)

- Checking the same pair of faces repeatedly.

# Our algorithm

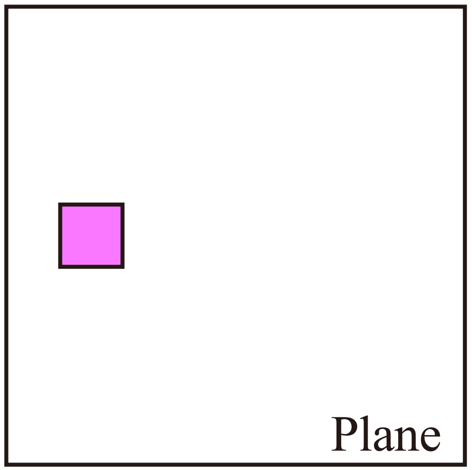
## Rotational Unfolding

1. Enumerating the path between any two faces by rolling a polyhedron in a "Koro Koro" approach.
2. Checking the overlap of both end-faces of a path.

# Our algorithm

## Rotational Unfolding

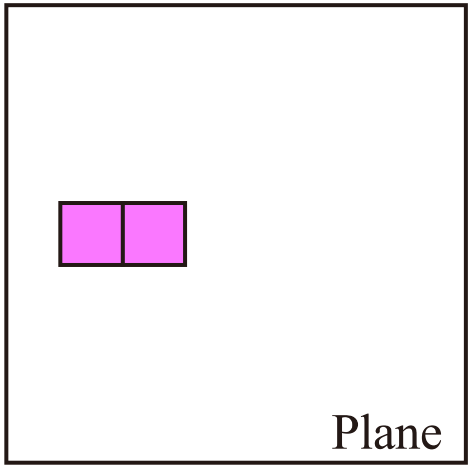
1. Enumerating the path between any two faces by rolling a polyhedron in a "Koro Koro" approach.
2. Checking the overlap of both end-faces of a path.

# Our algorithm

## Rotational Unfolding

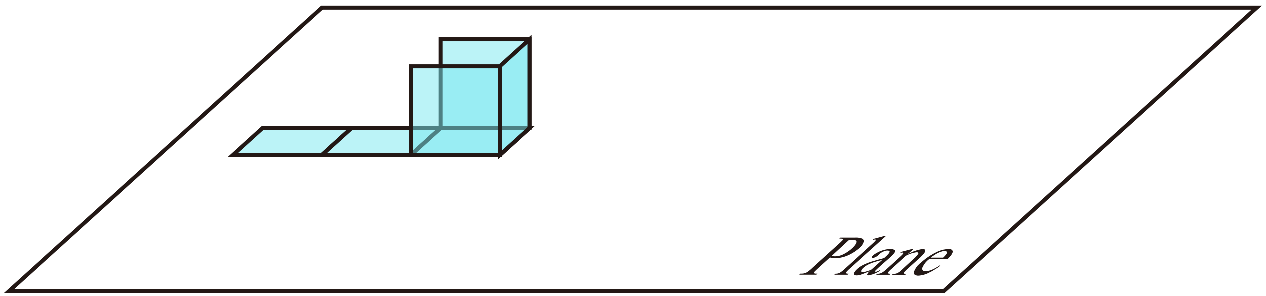
1. Enumerating the path between any two faces by rolling a polyhedron in a “Koro Koro” approach.
2. Checking the overlap of both end-faces of a path.

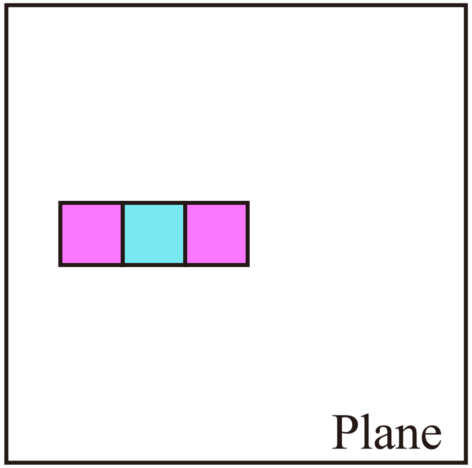



# Our algorithm

## Rotational Unfolding

1. Enumerating the path between any two faces by rolling a polyhedron in a “Koro Koro” approach.
2. Checking the overlap of both end-faces of a path.

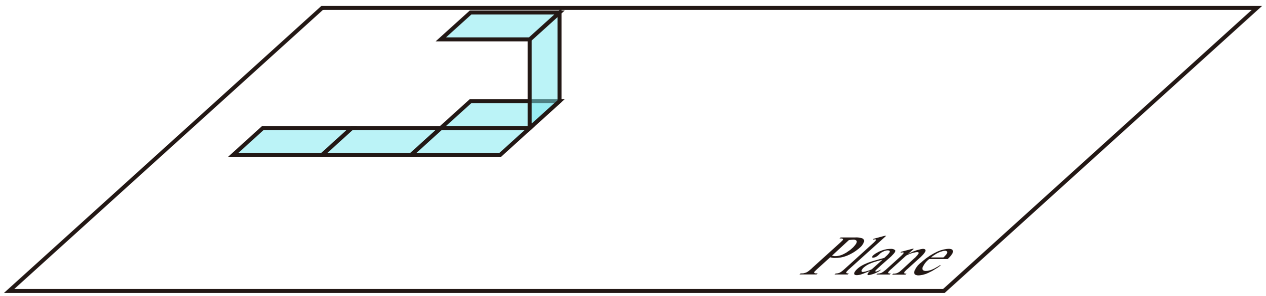


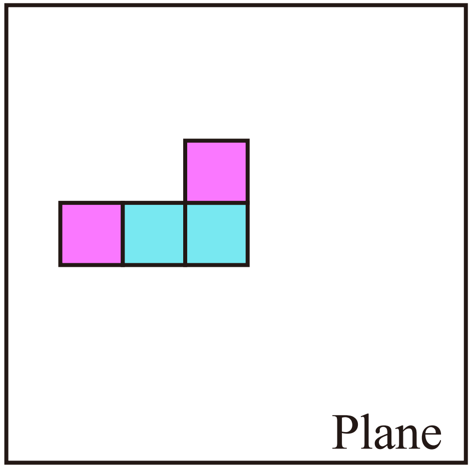



# Our algorithm

## Rotational Unfolding

1. Enumerating the path between any two faces by rolling a polyhedron in a "Koro Koro" approach.
2. Checking the overlap of both end-faces of a path.

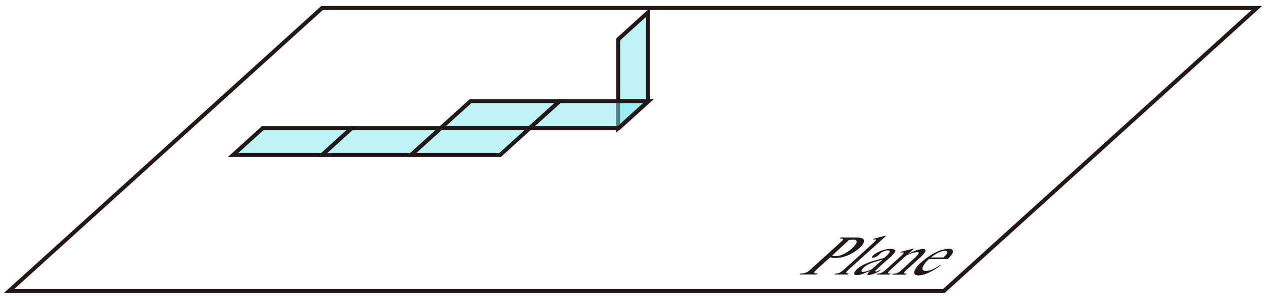


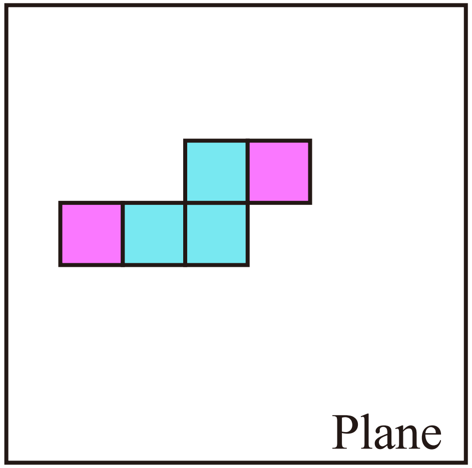



# Our algorithm

## Rotational Unfolding

1. Enumerating the path between any two faces by rolling a polyhedron in a "Koro Koro" approach.
2. Checking the overlap of both end-faces of a path.

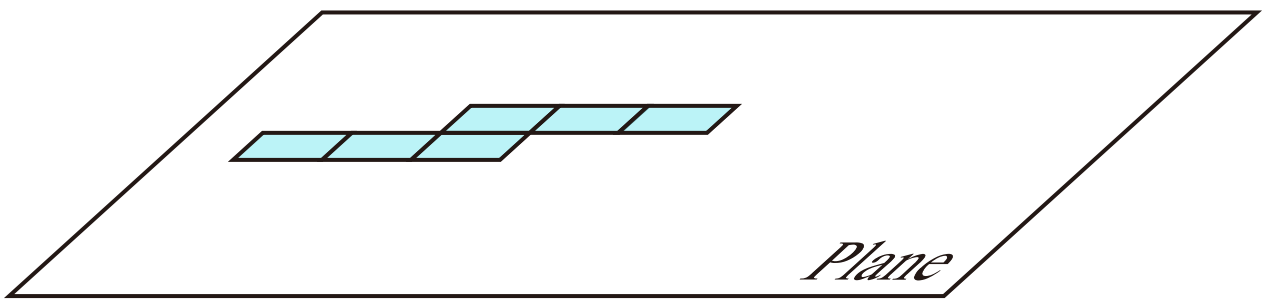


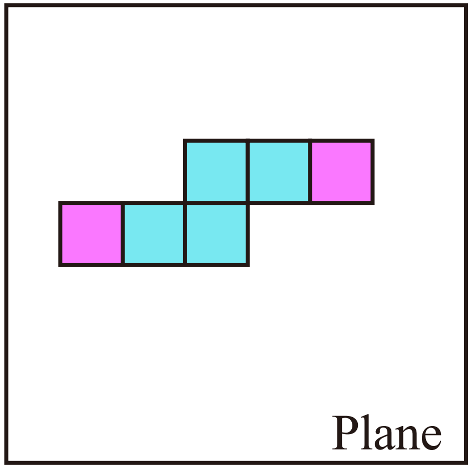



# Our algorithm

## Rotational Unfolding

1. Enumerating the path between any two faces by rolling a polyhedron in a "Koro Koro" approach.
2. Checking the overlap of both end-faces of a path.

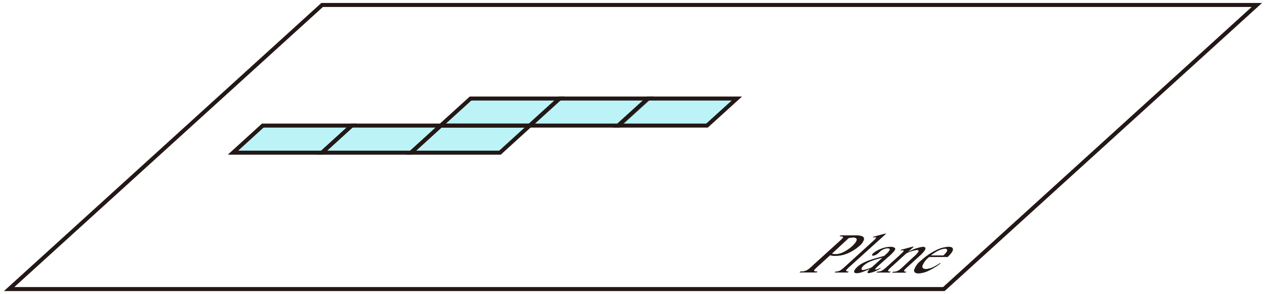


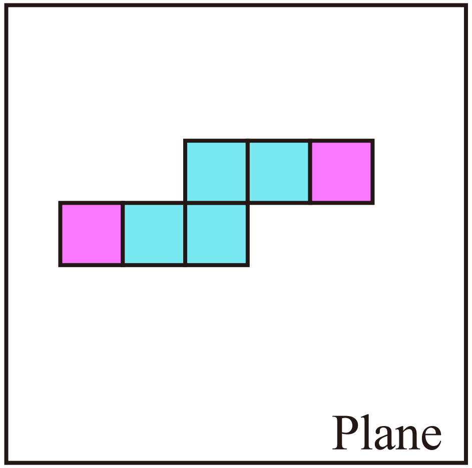



# Our algorithm

## Rotational Unfolding

1. Enumerating the path between any two faces by rolling a polyhedron in a "Koro Koro" approach.
2. Checking the overlap of both end-faces of a path.



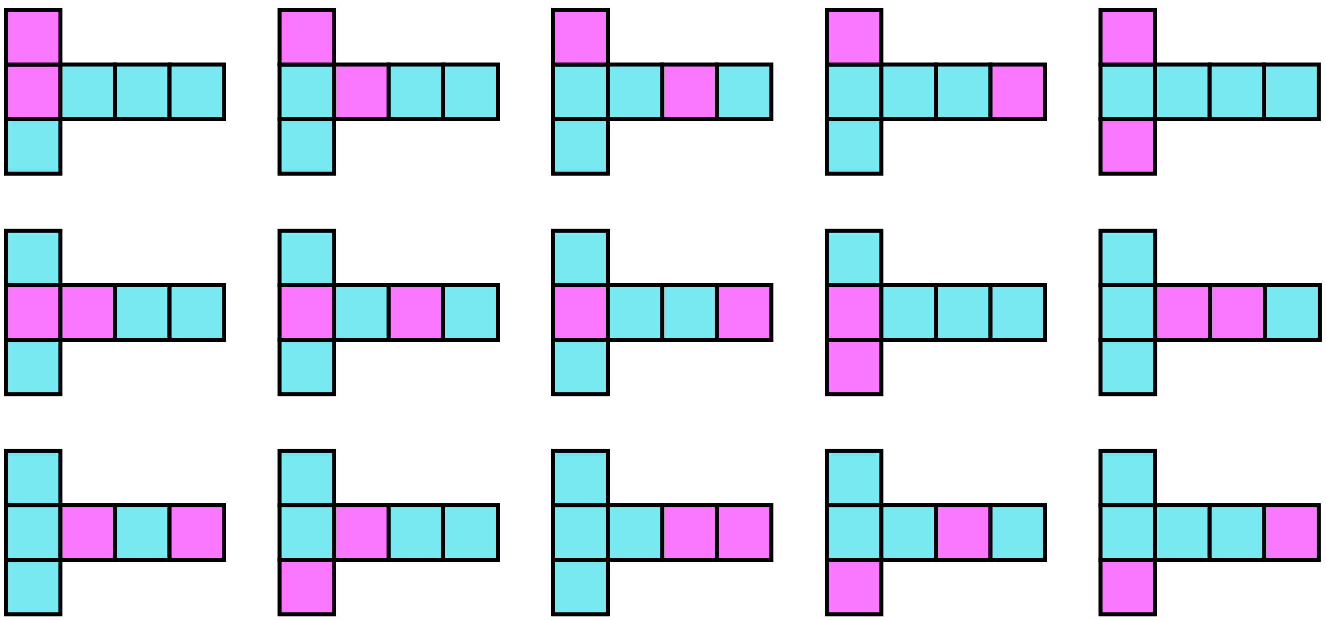

**Q.** Why only check the overlap of both end-faces in the path?

# Our algorithm

**Lemma 1**

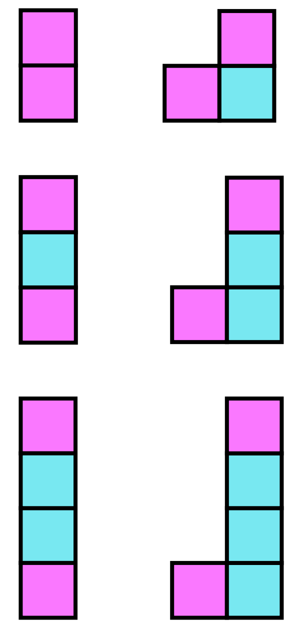
The path in the edge unfolding that connects two faces is one of the paths enumerated by rotational unfolding.

[T. Horiyama and W. Shoji, 2011]



${}_6C_2 = 15$ ways

Proposed



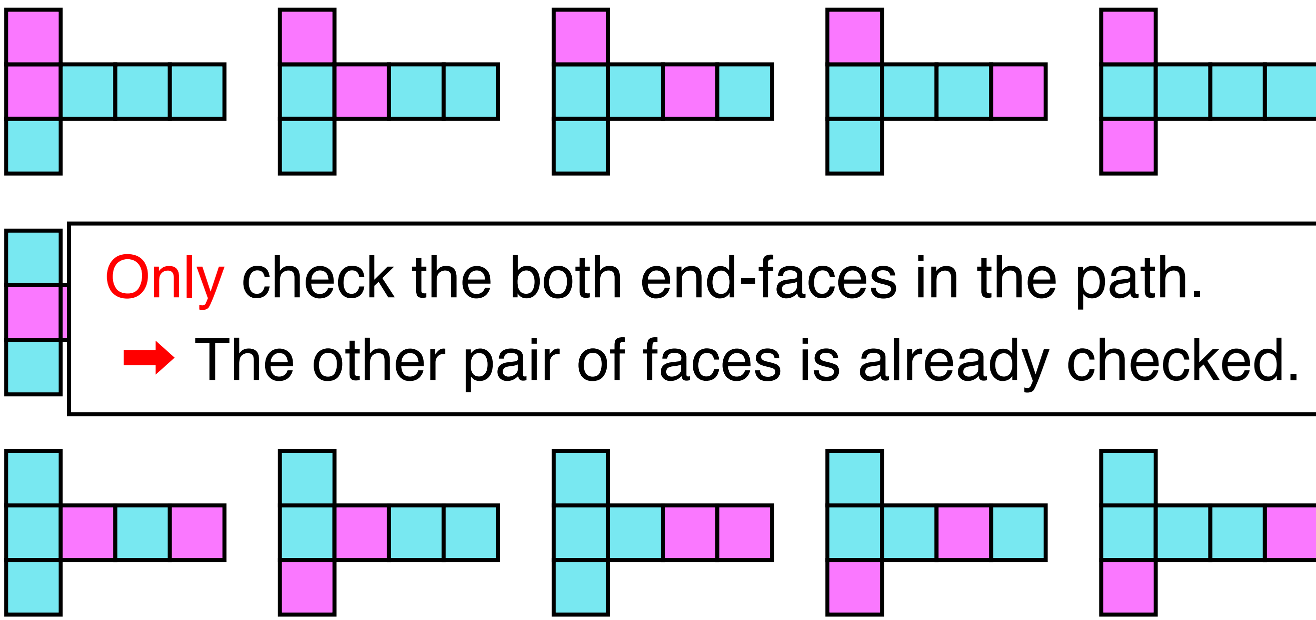
6 ways

# Our algorithm

**Lemma 1**

The path in the edge unfolding that connects two faces is one of the paths enumerated by rotational unfolding.

[T. Horiyama and W. Shoji, 2011]

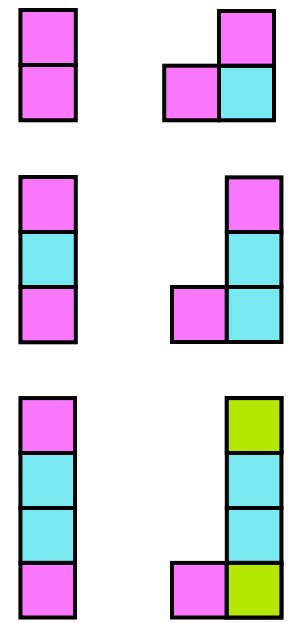


Only check the both end-faces in the path.

➡ The other pair of faces is already checked.

$_6C_2 = 15$ ways

Proposed

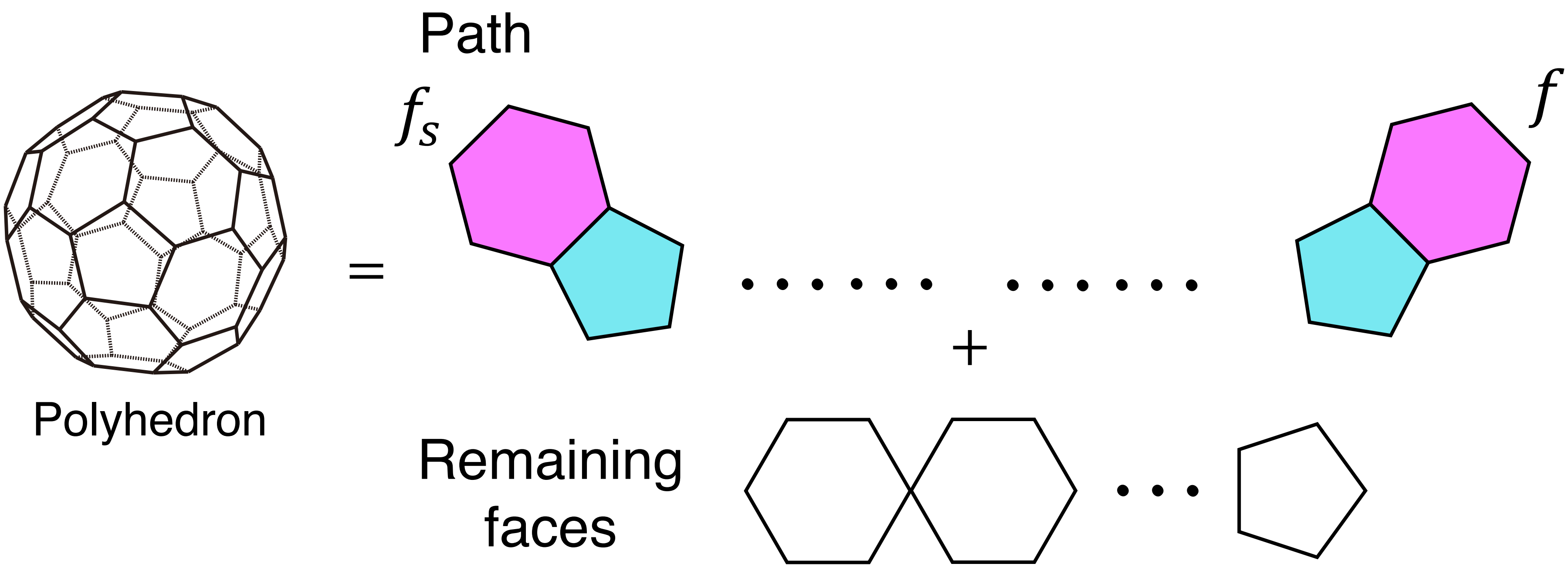


6 ways

# Ideas for more speed up

## Method 1

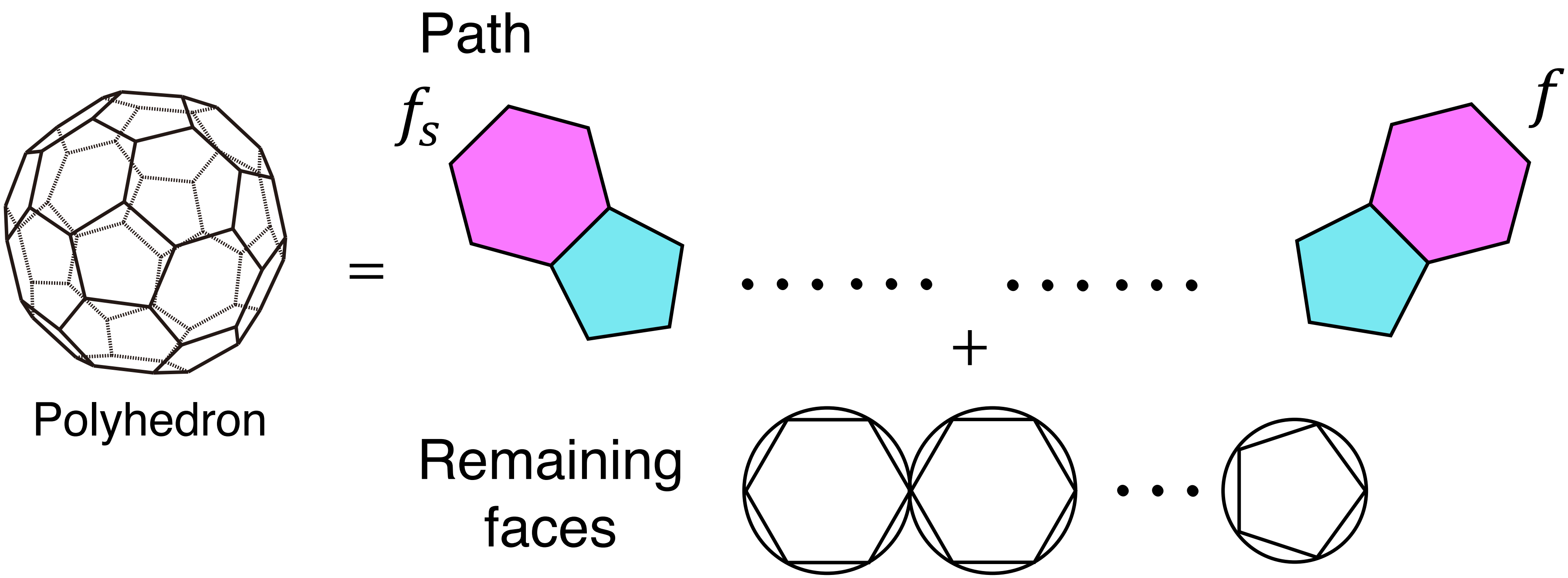
If the distance between the two end faces of a path is too far, we prune the search.



# Ideas for more speed up

**Method 1**

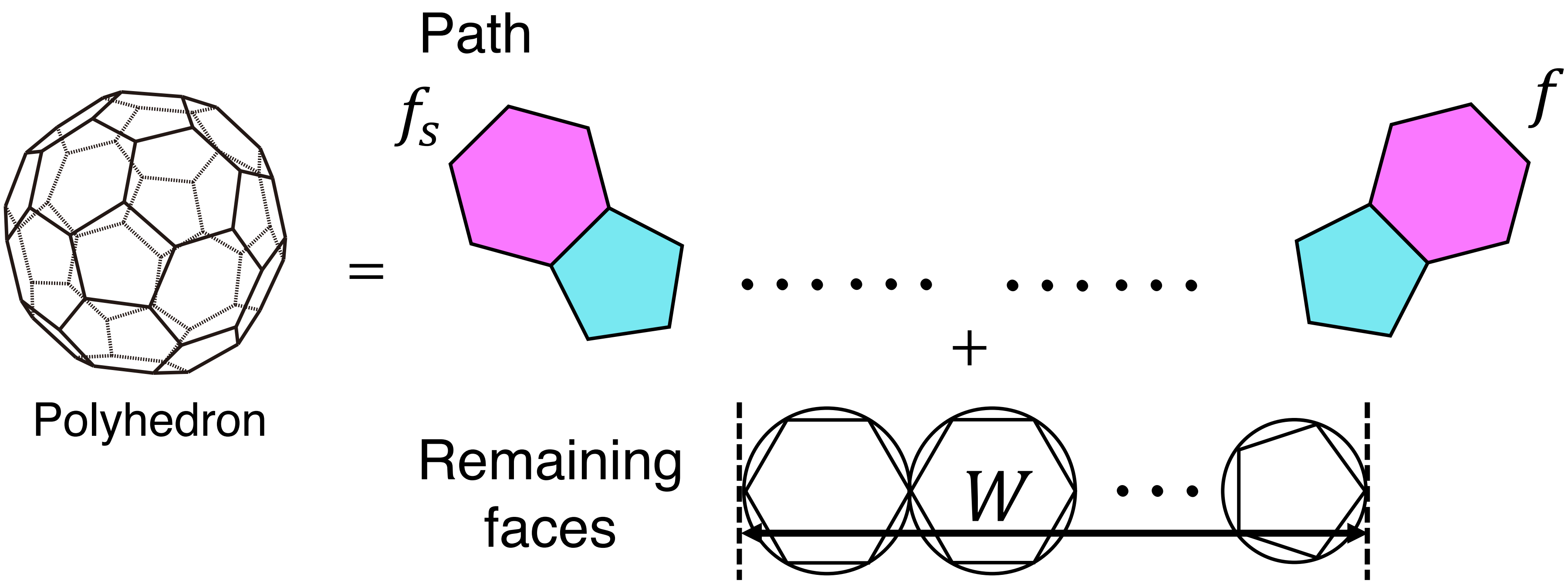
If the distance between the two end faces of a path is too far, we prune the search.



# Ideas for more speed up

## Method 1

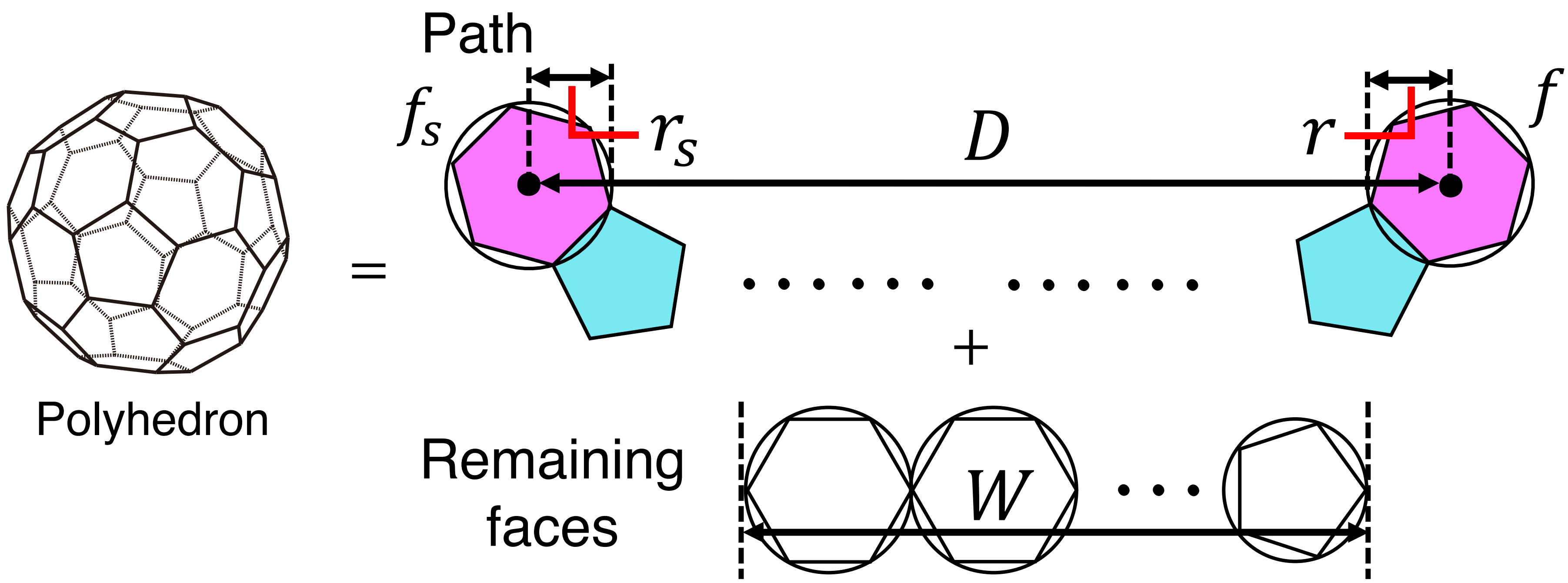
If the distance between the two end faces of a path is too far, we prune the search.



# Ideas for more speed up

## Method 1

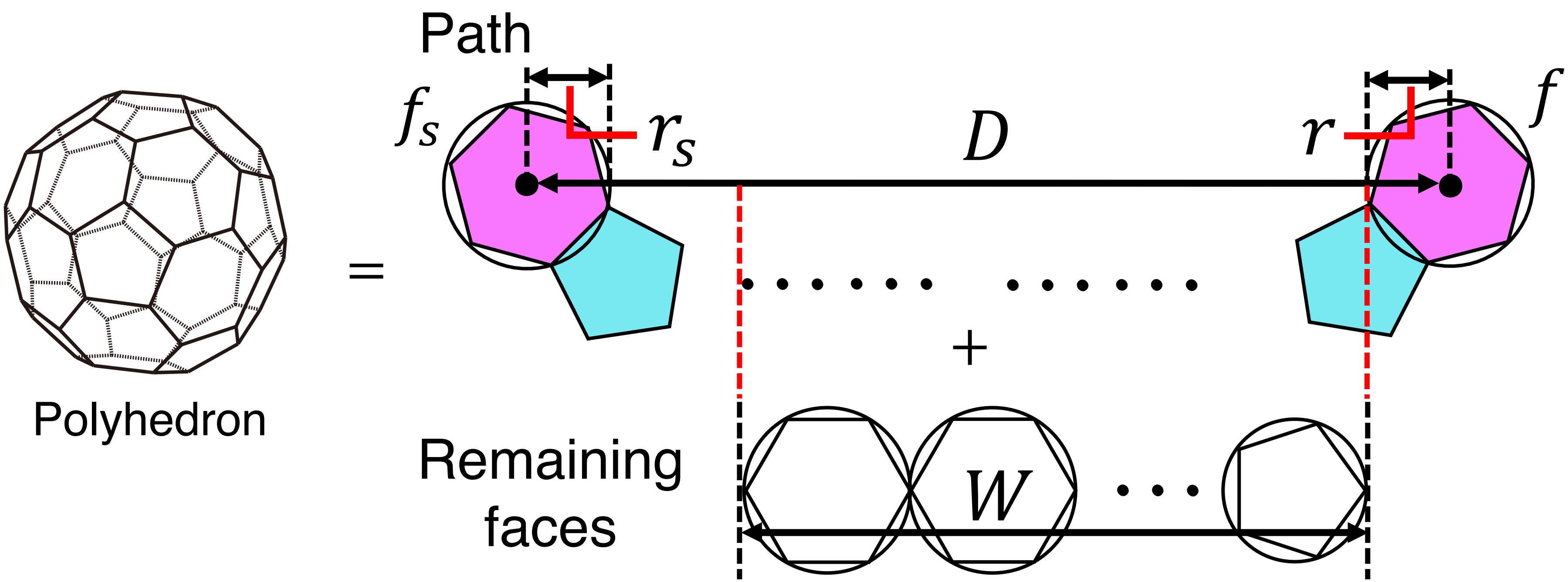
If the distance between the two end faces of a path is too far, we prune the search.



# Ideas for more speed up

## Method 1

If the distance between the two end faces of a path is too far, we prune the search.

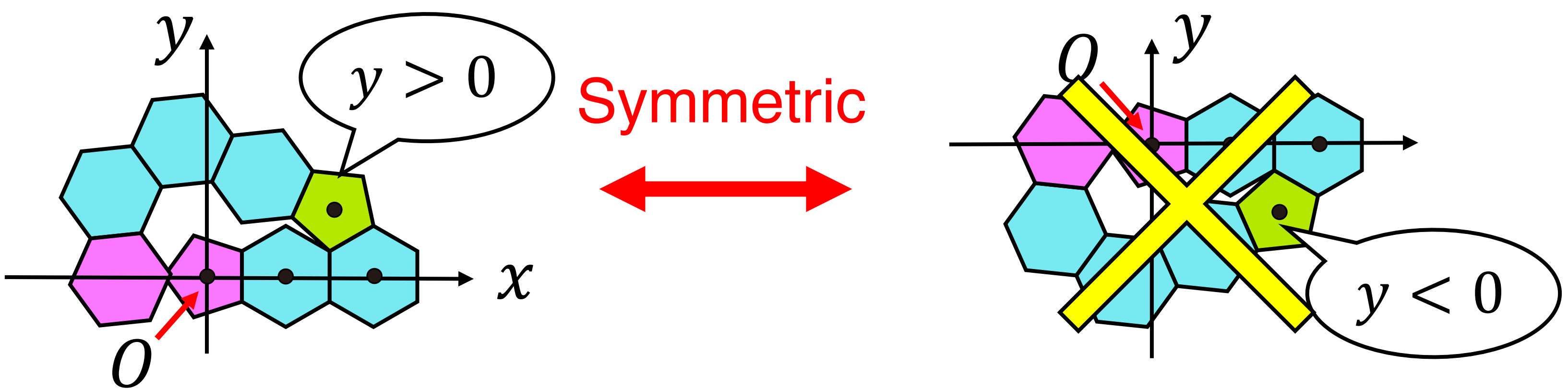



If $W < D - r_s - r$, $f_s$ does not overlap.

# Ideas for more speed up

**Method 2**

If a polyhedron has symmetric unfoldings, we only compute one of them.



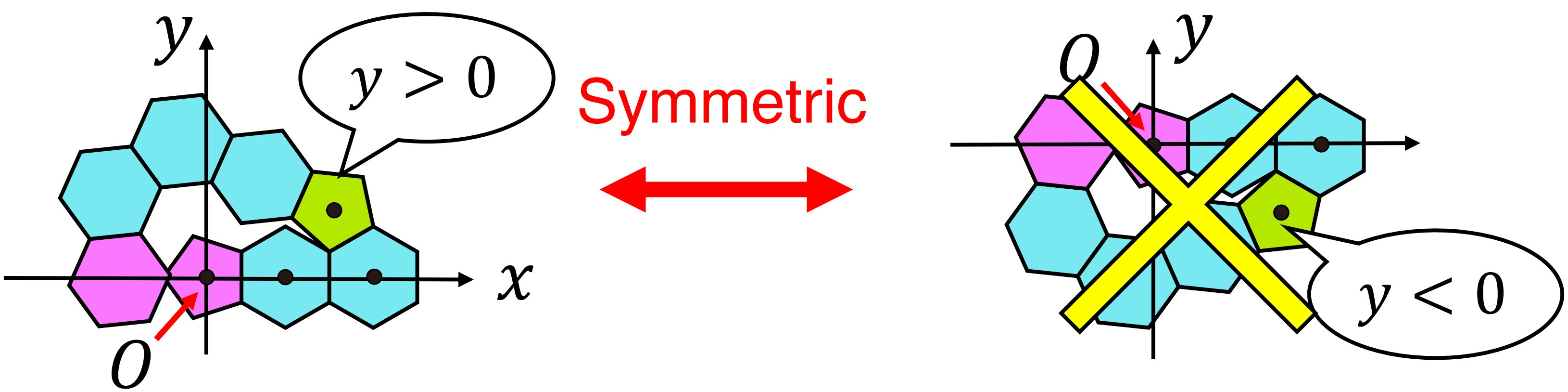

The $y$-coordinate becomes …

① Non-zero for the first time ② Negative ➡ Prune the search

# Ideas for more speed up

**Method 2**

If a polyhedron has symmetric unfoldings, we only compute one of them.




The $y$-coordinate becomes …

① Non-zero for the first time ② Negative ➡ Prune the search

Note : This pruning does not work for non-symmetry polyhedron.

(Ex.) Snub cube, Snub dodecahedron and Johnson solids

# Summary

| Convex regular-faced polyhedra | Is there an overlapping edge unfolding? |
|---|---|
| Platonic solids (Total 5 types) | No [T. Horiyama and W. Shoji, 2011] |
| Archimedean solids (Total 13 types) | Yes (5 types) [T. Horiyama and W. Shoji, 2011]<br>No (5 types) [Hirose, 2015]<br>☆ **No (2 types), Yes (1 type)** |
| $n$-gonal Archimedean prisms ( $n \geq 3$ ) | ☆ **No** $(3 \leq n \leq 23)$<br>**Yes** $(n \geq 24)$ |
| $n$-gonal Archimedean antiprisms ( $n \geq 3$ ) | ☆ **No** $(3 \leq n \leq 11)$<br>**Yes** $(n \geq 12)$ |
| Johnson solids (Total 92 types) | ☆ **No (48 types)**<br>☆ **Yes (44 types)** |

# Summary

| Convex regular-faced polyhedra | Is there an overlapping edge unfolding? |
|---|---|
| Platonic solids (Total 5 types) | No [T. Horiyama and W. Shoji, 2011] |
| Archimedean solids (Total 13 types) | Yes (5 types) [T. Horiyama and W. Shoji, 2011]<br>No (5 types) [Hirose, 2015]<br>☆ **No (2 types), Yes (1 type)** |
| $n$-gonal Archimedean prisms ( $n \geq 3$ ) | ☆ **No** $(3 \leq n \leq 23)$<br>**Yes** $(n \geq 24)$ |
| $n$-gonal Archimedean antiprisms ( $n \geq 3$ ) | ☆ **No** $(3 \leq n \leq 11)$<br>**Yes** $(n \geq 12)$ |
| Johnson solids (Total 92 types) | ☆ **No (48 types)**<br>☆ **Yes (44 types)** |

# Summary

| Convex regular-faced polyhedra | Is there an overlapping edge unfolding? |
|---|---|
| Platonic solids (Total 5 types) | No [T. Horiyama and W. Shoji, 2011] |
| Archimedean solids (Total 13 types) | Yes (5 types) [T. Horiyama and W. Shoji, 2011]<br>No (5 types) [Hirose, 2015]<br>☆ **No (2 types), Yes (1 type)** |
| $n$-gonal Archimedean prisms ( $n \geq 3$ ) | ☆ **No ($3 \leq n \leq 23$)**<br>**Yes ($n \geq 24$)** |
| $n$-gonal Archimedean antiprisms ( $n \geq 3$ ) | ☆ **No ($3 \leq n \leq 11$)**<br>**Yes ($n \geq 12$)** |
| Johnson solids (Total 92 types) | ☆ **No (48 types)**<br>☆ **Yes (44 types)** |

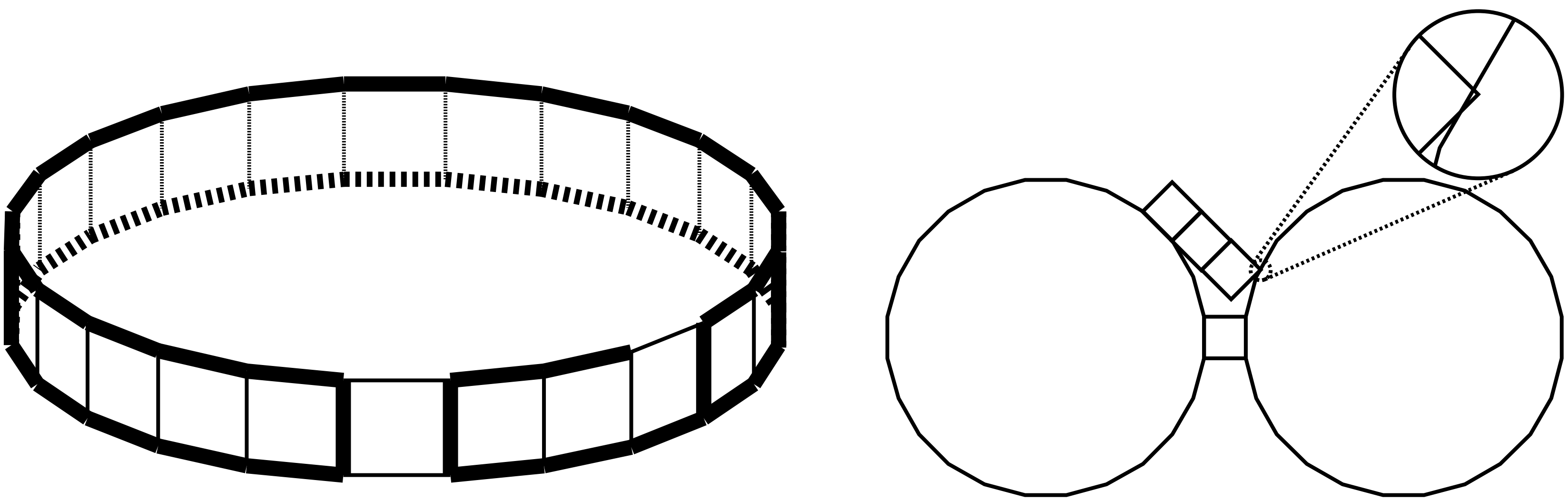
**Fully proved the existence!**

# Proof of Theorem 2

Theorem 2 (Restated)

2. Overlapping edge unfoldings exist in $n$-gonal Archimedean prisms for $n \geq 24$.

**Proof.**



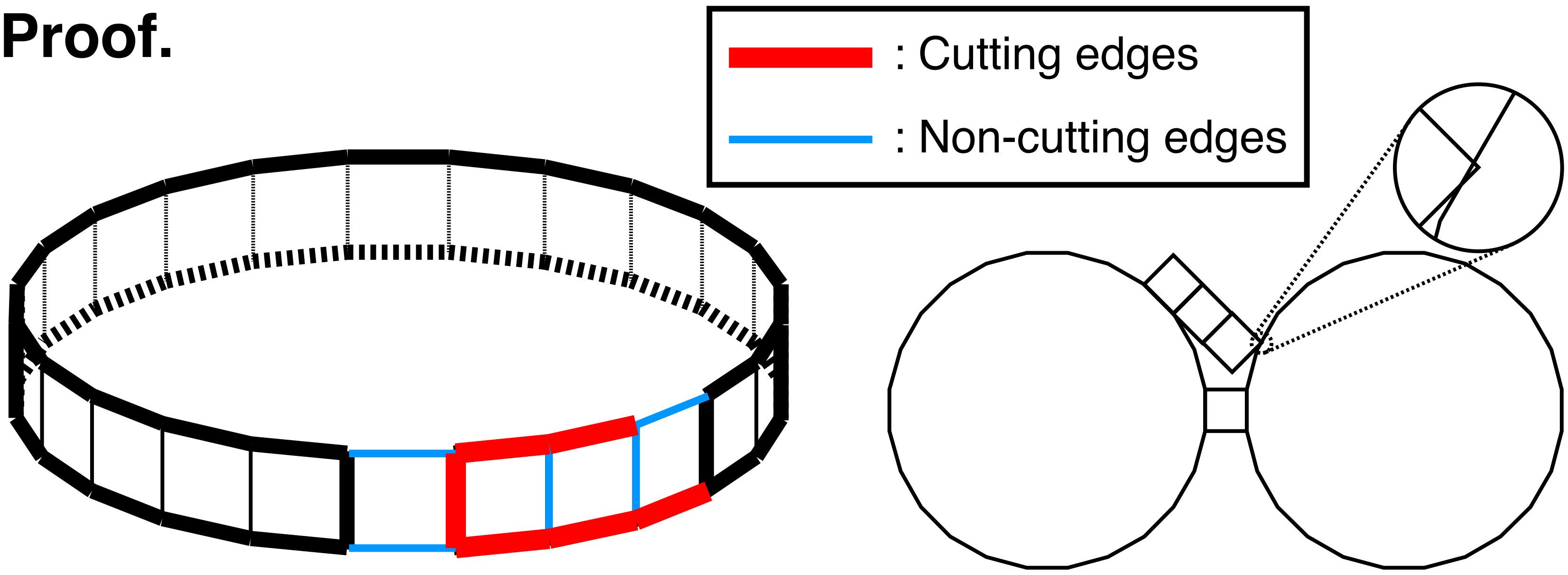
An overlapping edge unfolding in a $24$-gonal Archimedean prism

# Proof of Theorem 2

Theorem 2 (Restated)

2. Overlapping edge unfoldings exist in $n$-gonal Archimedean prisms for $n \geq 24$.

**Proof.**

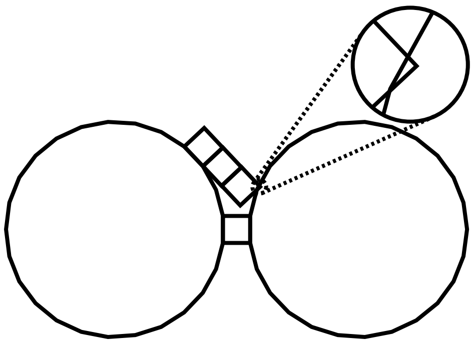



An overlapping edge unfolding in a $24$-gonal Archimedean prism

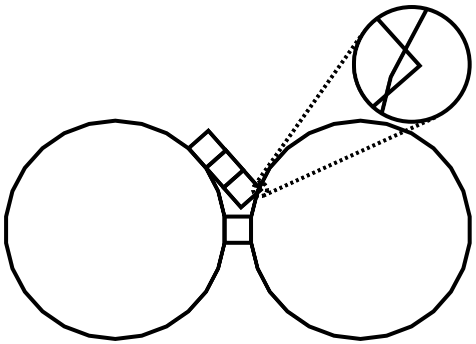
# Proof of Theorem 2

**Proof (continued).**

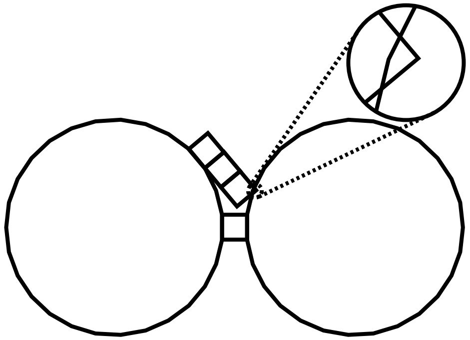
- Cutting edges / Non-cutting edges be the same as $n = 24$



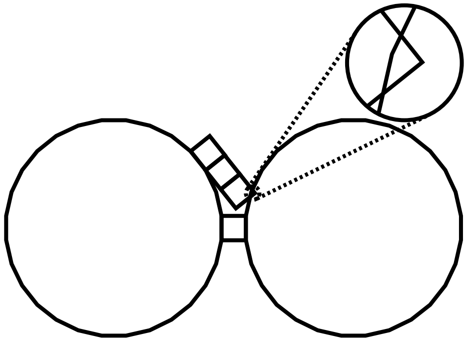
25-gonal Archimedean prism



26-gonal Archimedean prism



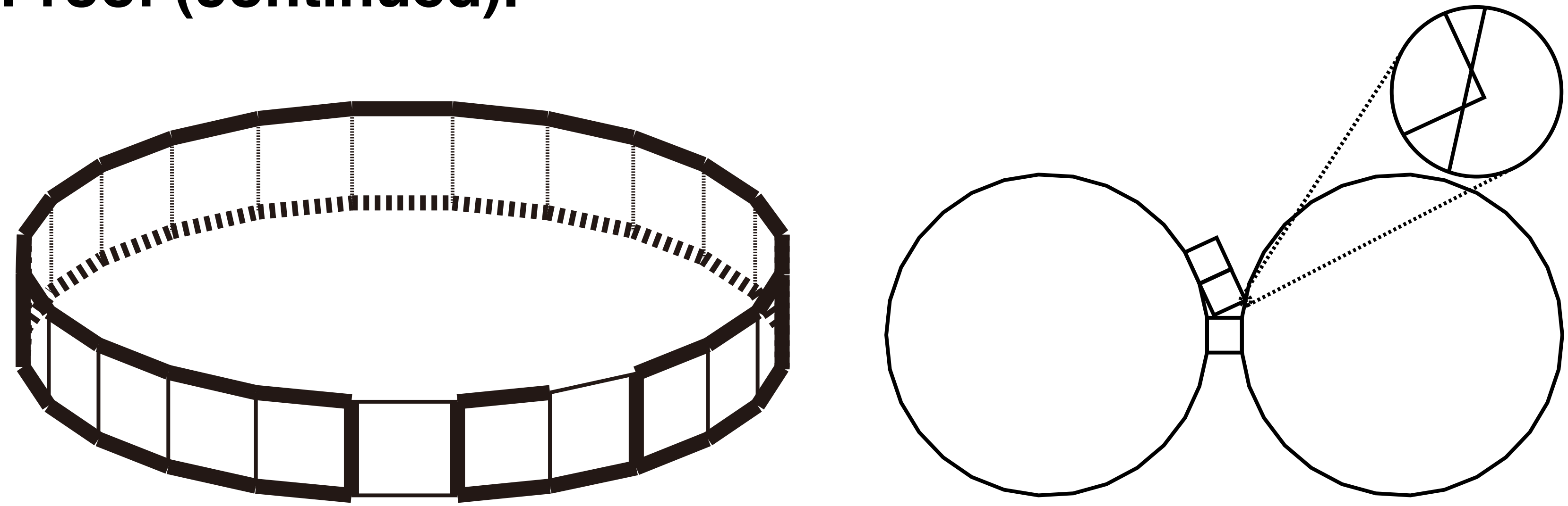
27-gonal Archimedean prism



28-gonal Archimedean prism

# Proof of Theorem 2

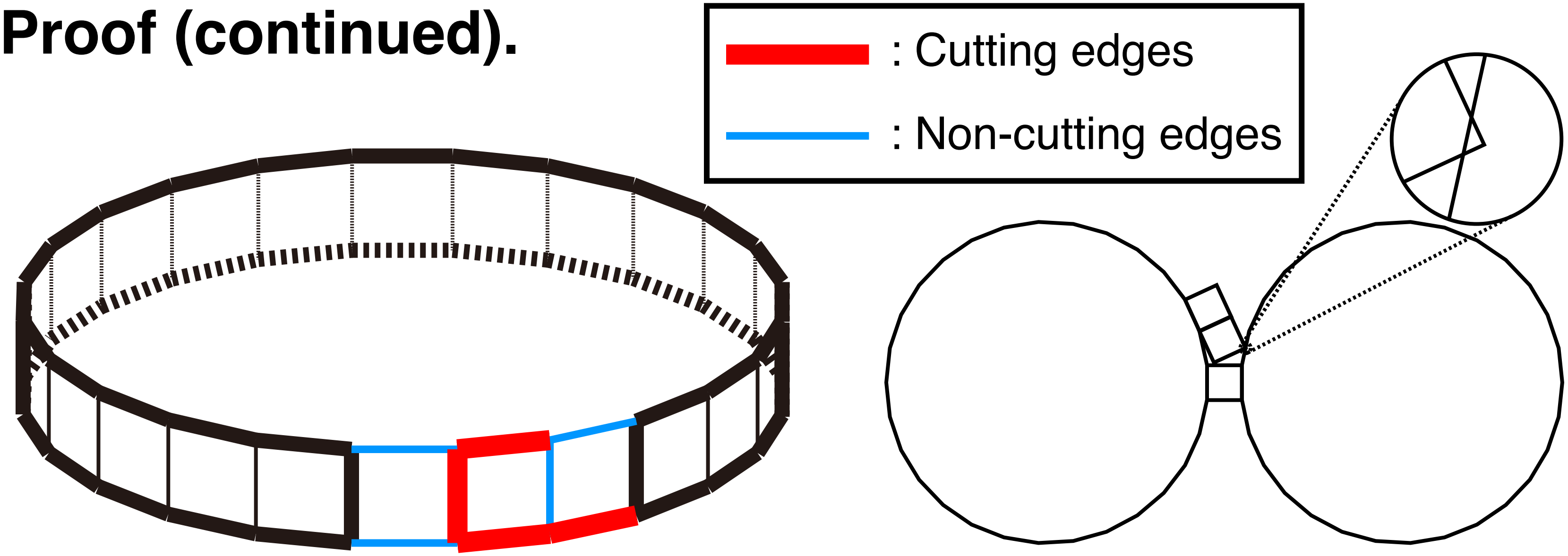
**Proof (continued).**



An overlapping edge unfolding in a $29$-gonal Archimedean prism

# Proof of Theorem 2

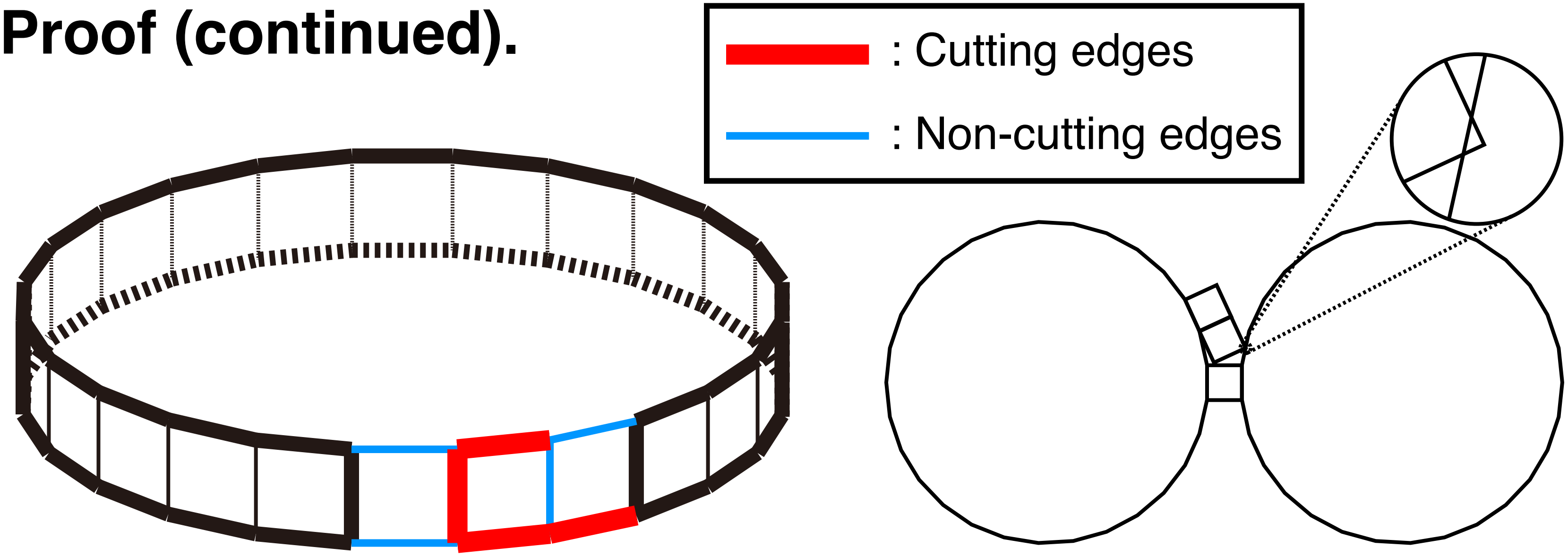
**Proof (continued).**




An overlapping edge unfolding in a $29$-gonal Archimedean prism

# Proof of Theorem 2

**Proof (continued).**




An overlapping edge unfolding in a $29$-gonal Archimedean prism

**Lemma 2**

Overlapping edge unfoldings exist in $n$-gonal Archimedean prisms for $n \geq 29$.

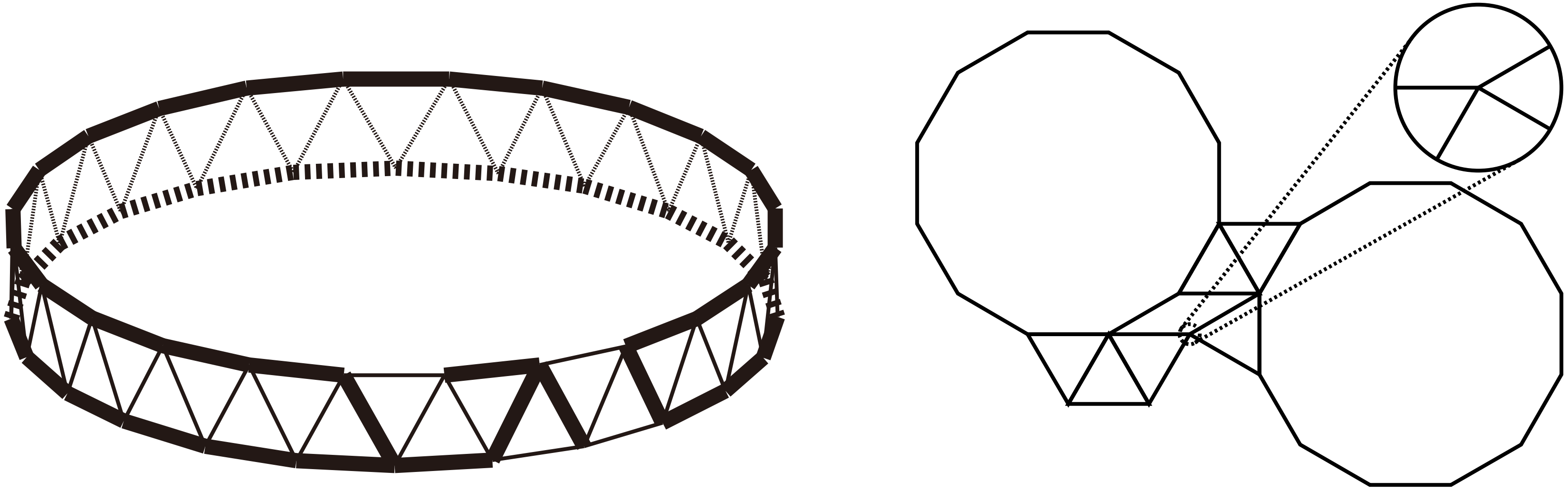
Cutting / non-cutting edges be the same as $n = 29$. ■

# Proof of Theorem 3

Theorem 3 (Restated)

2. Overlapping edge unfoldings exist in $n$-gonal Archimedean antiprisms for $n \geq 12$.

**Proof.**



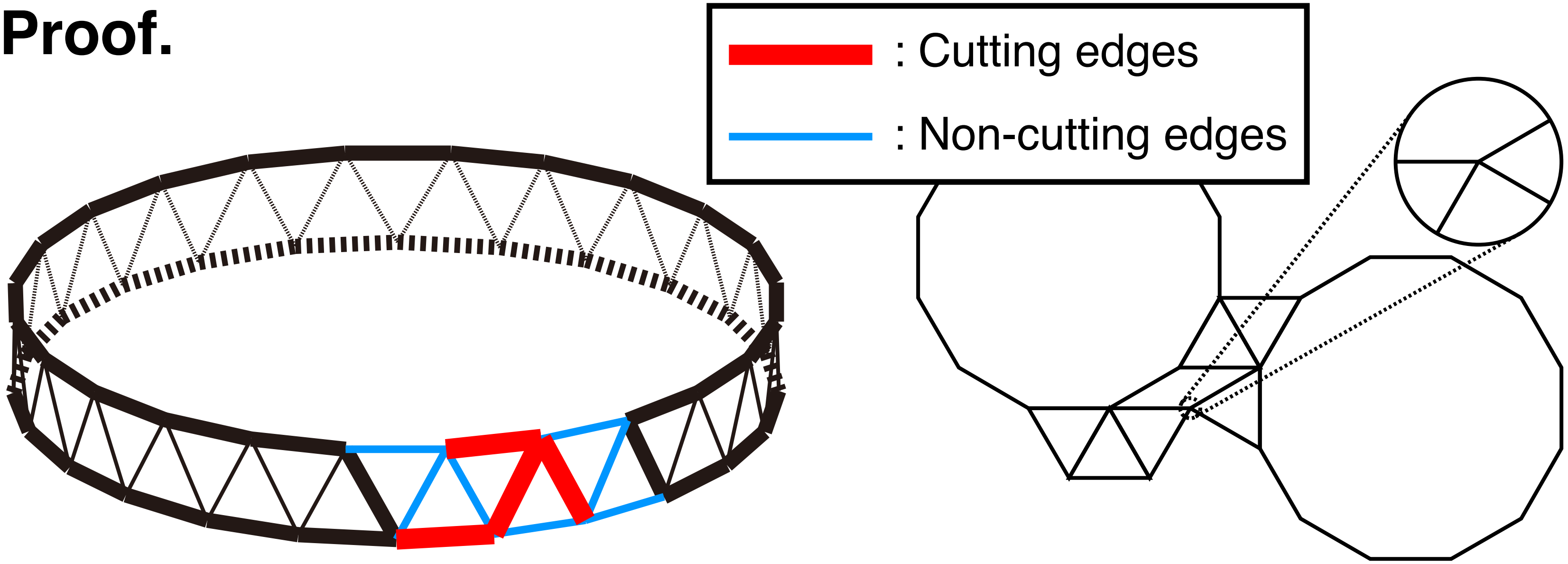
An overlapping edge unfolding in a $12$-gonal Archimedean antiprism

# Proof of Theorem 3

**Theorem 3 (Restated)**

2. Overlapping edge unfoldings exist in $n$-gonal Archimedean antiprisms for $n \geq 12$.

**Proof.**

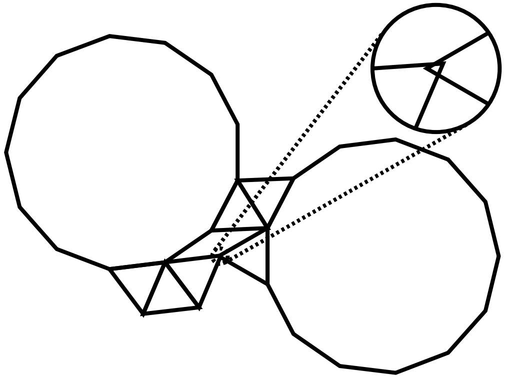



An overlapping edge unfolding in a $12$-gonal Archimedean antiprism

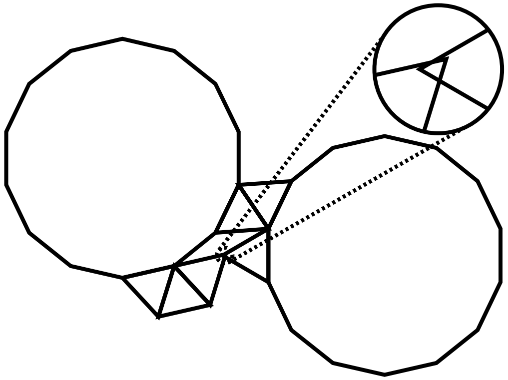
# Proof of Theorem 3

**Proof (continued).**

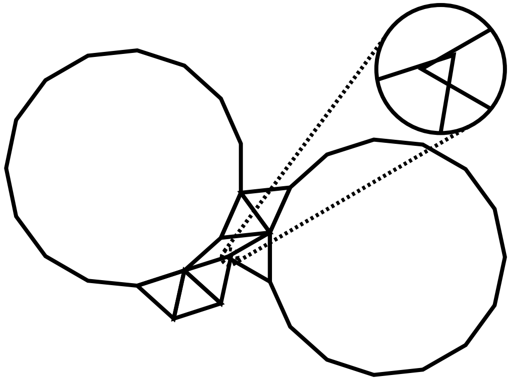
- Cutting edges / Non-cutting edges be the same as $n = 12$.



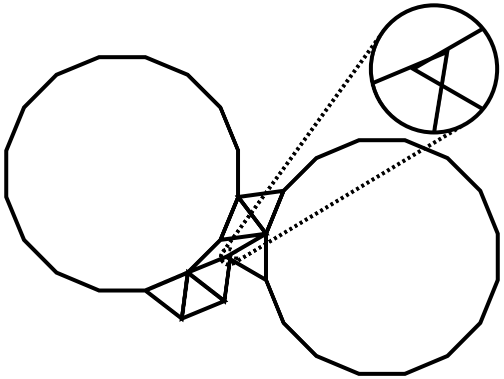
$13$-gonal Archimedean antiprism



$14$-gonal Archimedean antiprism



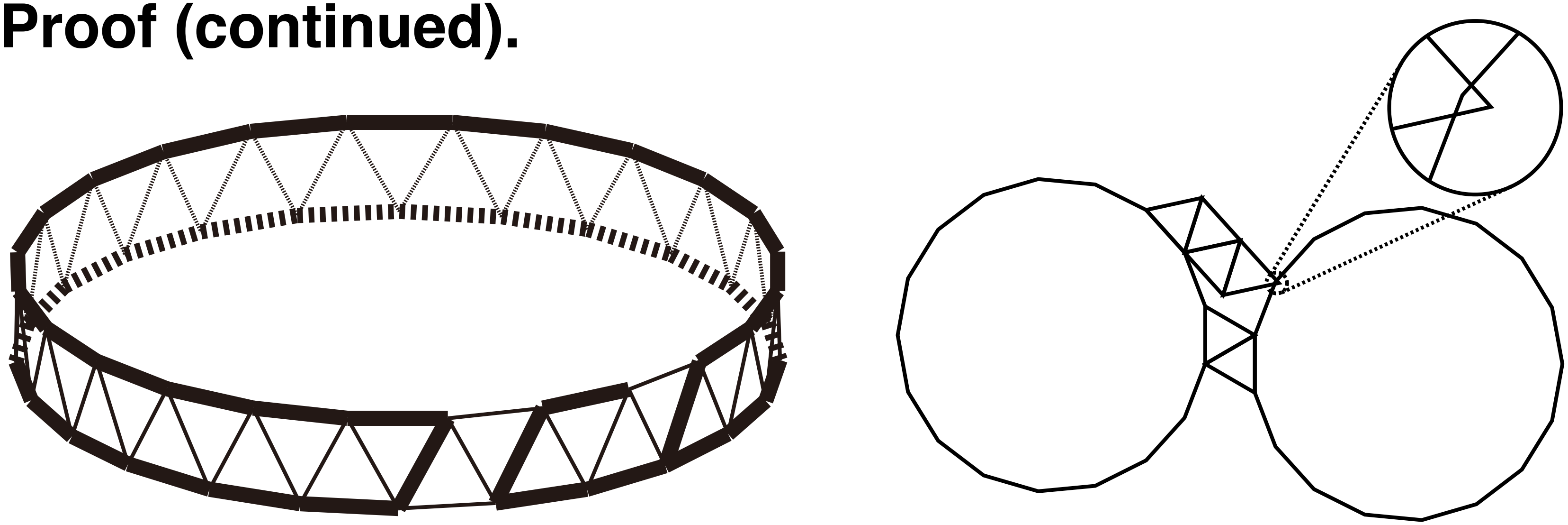
$15$-gonal Archimedean antiprism



$16$-gonal Archimedean antiprism

# Proof of Theorem 3

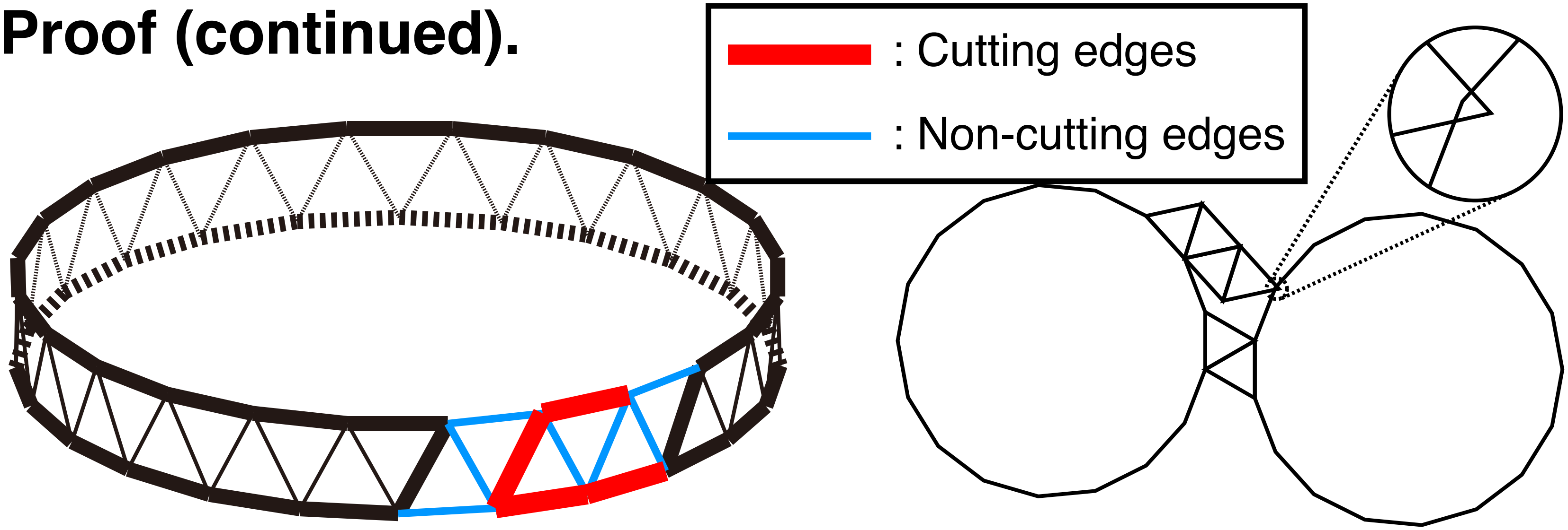
**Proof (continued).**



An overlapping edge unfolding in a $17$-gonal Archimedean antiprism

# Proof of Theorem 3

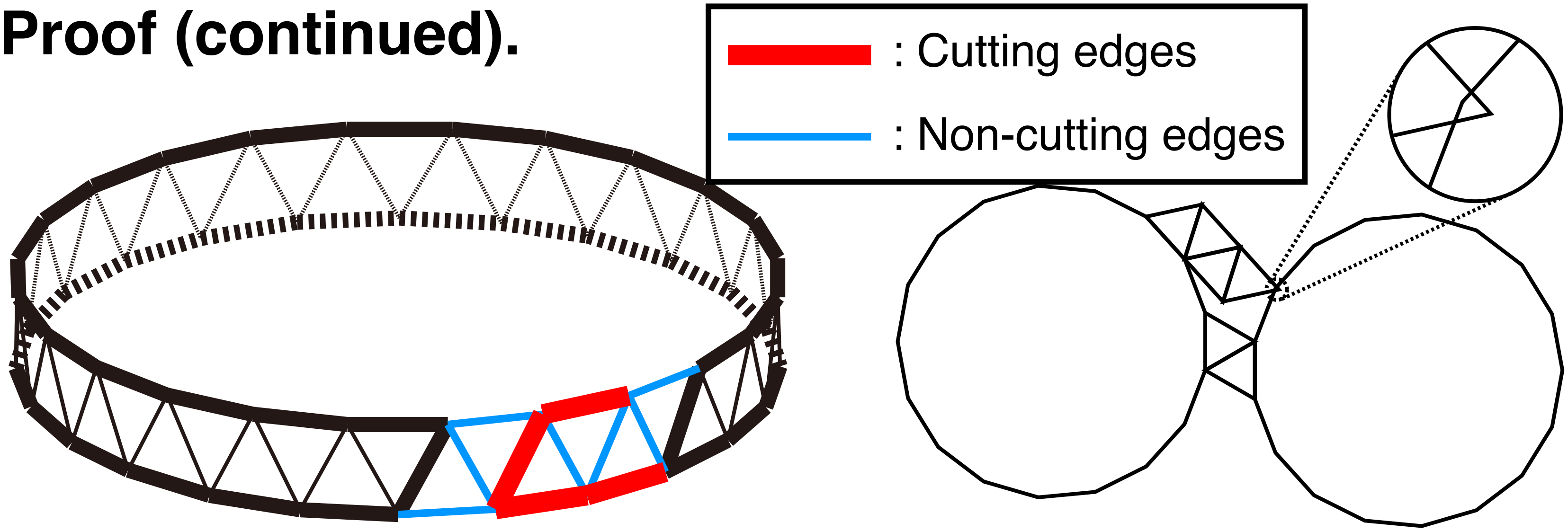
**Proof (continued).**




An overlapping edge unfolding in a 17-gonal Archimedean antiprism

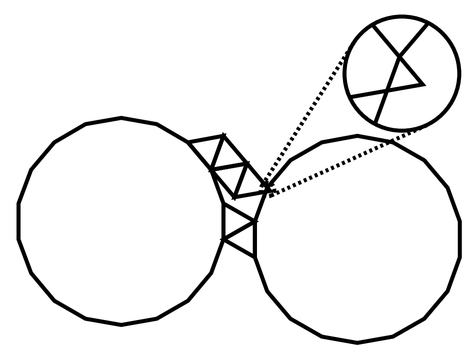
# Proof of Theorem 3

**Proof (continued).**




An overlapping edge unfolding in a $17$-gonal Archimedean antiprism

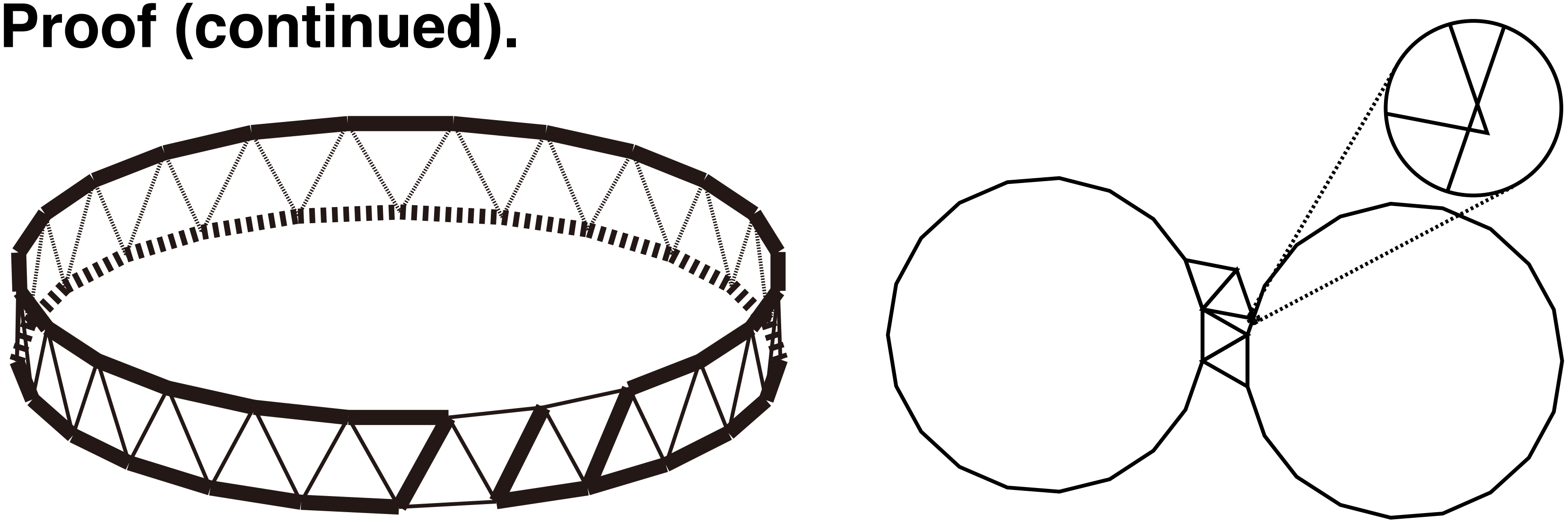
- Cutting / non-cutting edges be the same as $n = 17$.



$18$-gonal Archimedean antiprism

# Proof of Theorem 3

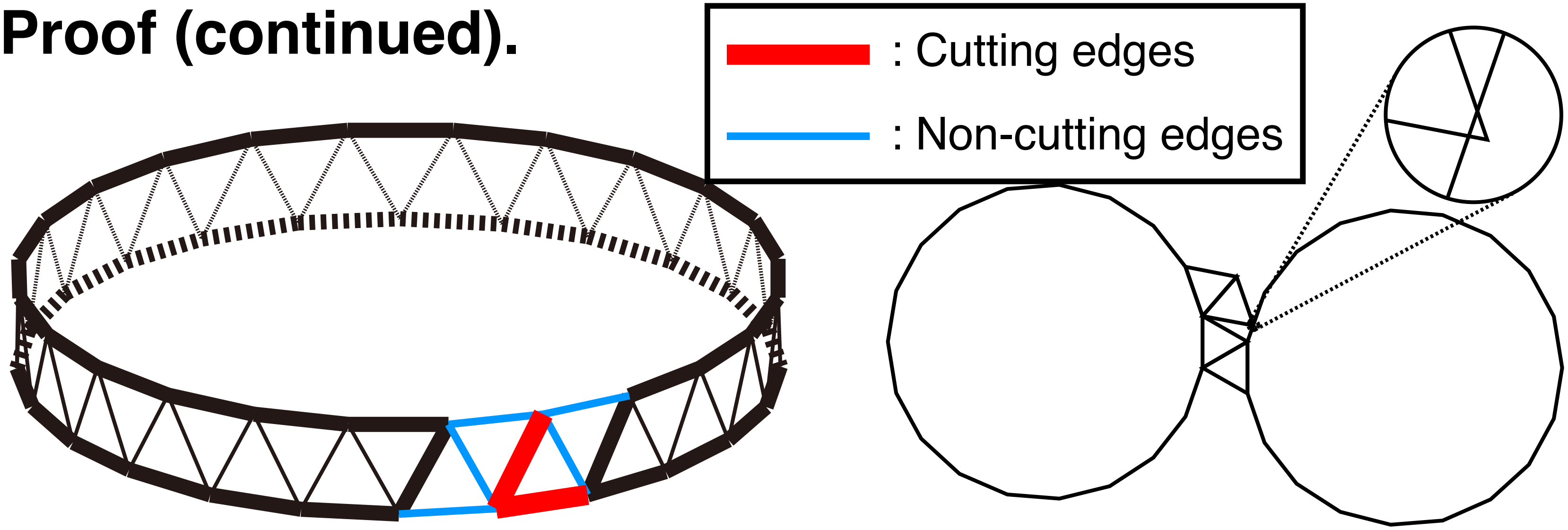
**Proof (continued).**



An overlapping edge unfolding in a $19$-gonal Archimedean antiprism

# Proof of Theorem 3

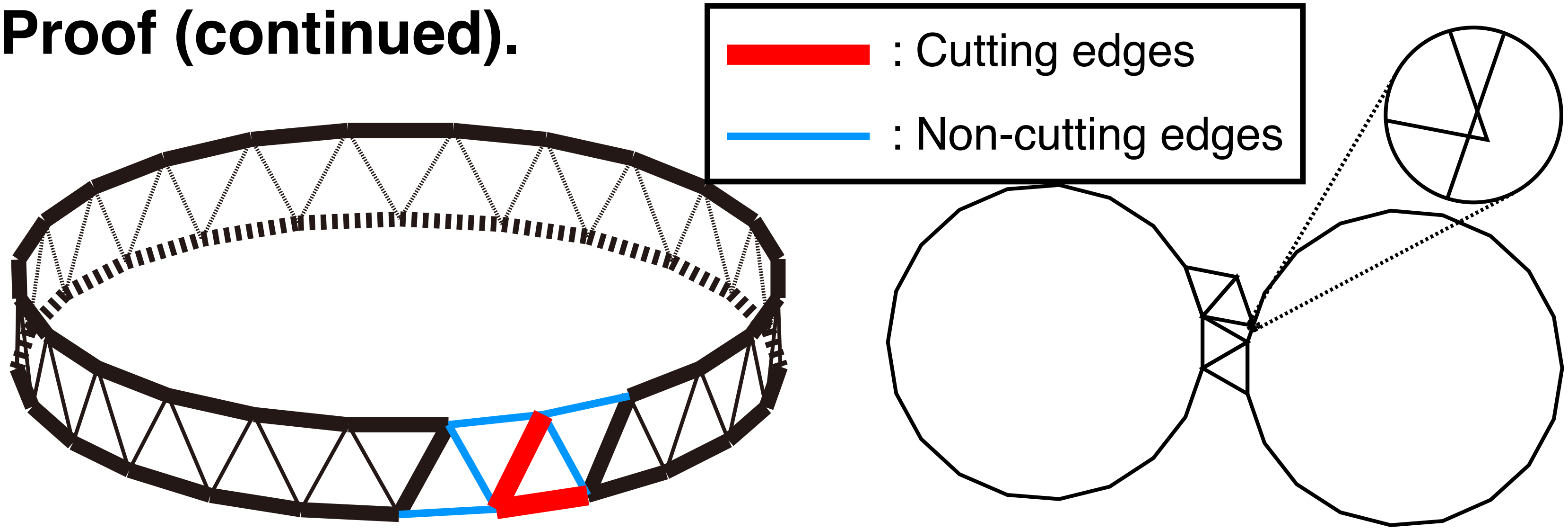
**Proof (continued).**




An overlapping edge unfolding in a $19$-gonal Archimedean antiprism

# Proof of Theorem 3

**Proof (continued).**




An overlapping edge unfolding in a $19$-gonal Archimedean antiprism

**Lemma 3**

Overlapping edge unfoldings exist in $n$-gonal Archimedean antiprisms for $n \geq 19$.

Cutting / non-cutting edges be the same as $n = 19$. ■