Overlapping of Lattice Unfolding for Cuboids

CCCG 2023

◎ Takumi SHIOTA[†], Tonan KAMATA[‡], Ryuhei UEHARA[‡]

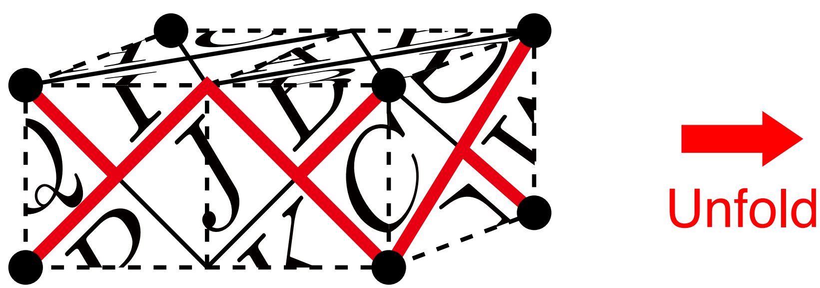
[†] Kyushu Institute of Technology, Japan
[‡] Japan Advanced Institute of Science and Technology, Japan

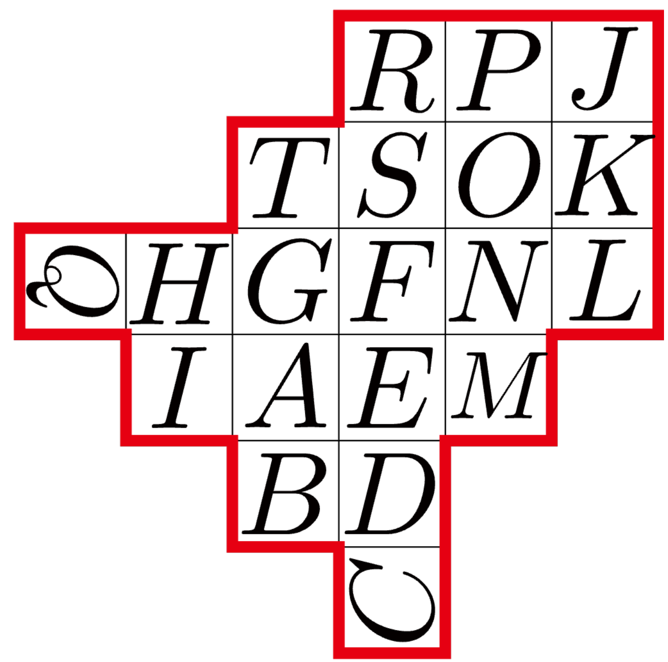
August 2, 2023

Overlapping of lattice unfolding

Let's consider unfolding a cuboid into a polyomino.

[Note] A *polyomino* is a polygon made by connecting multiple squares along their edges.





- Let's call this type of polyomino "Lattice unfolding".

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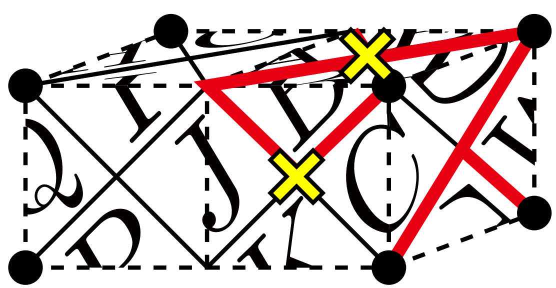
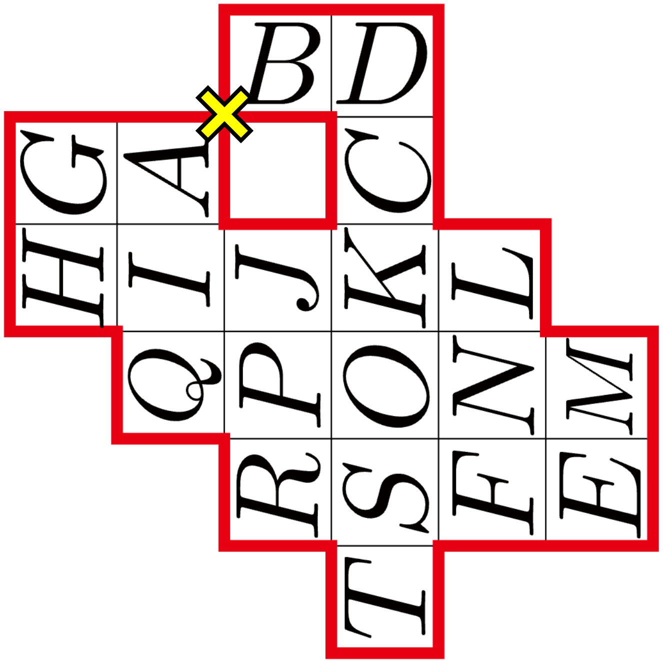


Unfold



- We call this type of unfolding "Vertices-in-touch unfolding".

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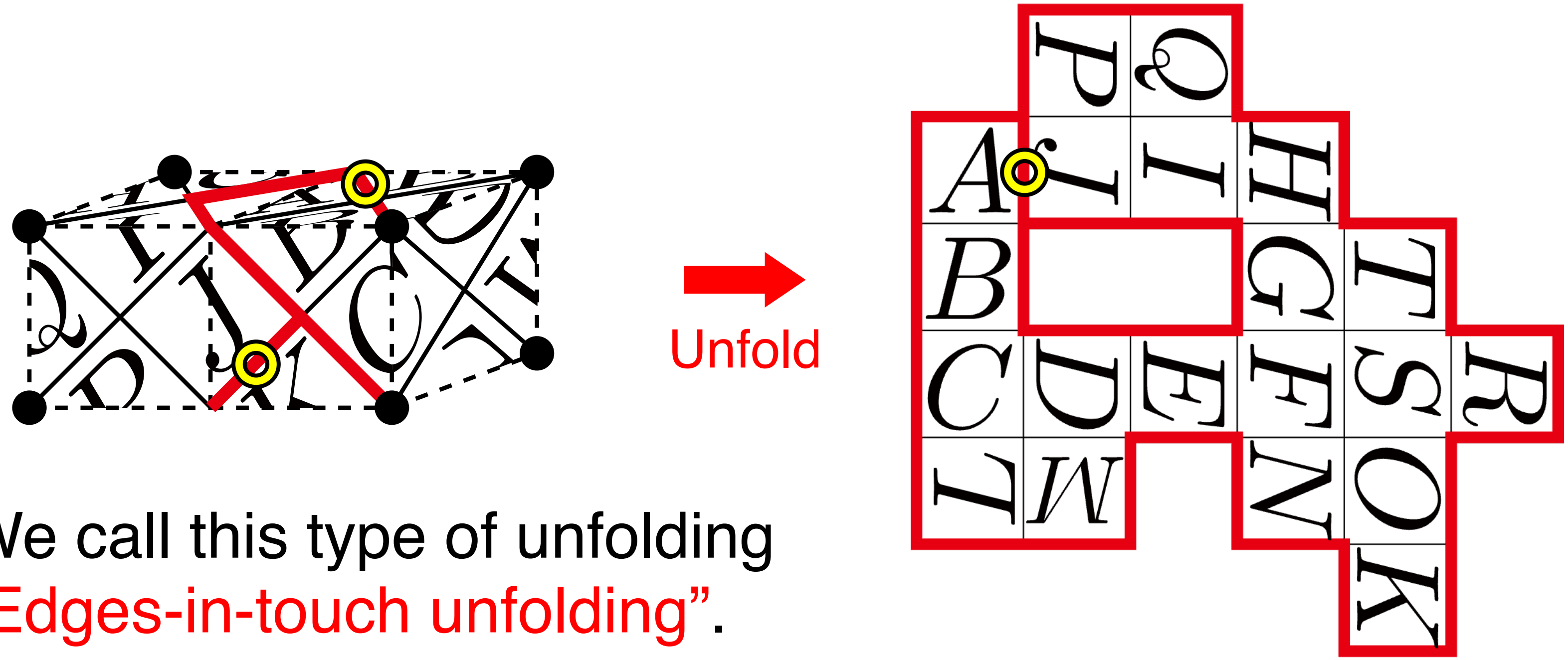

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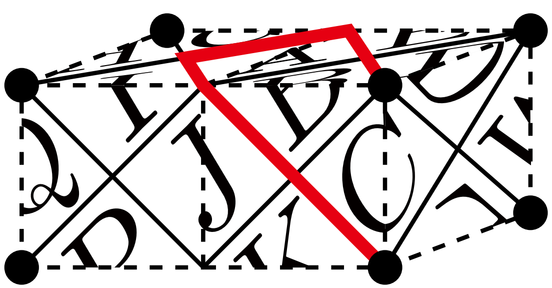

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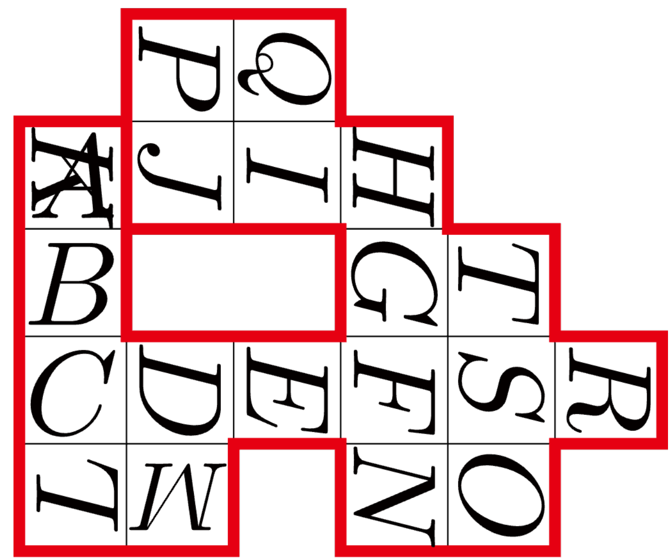


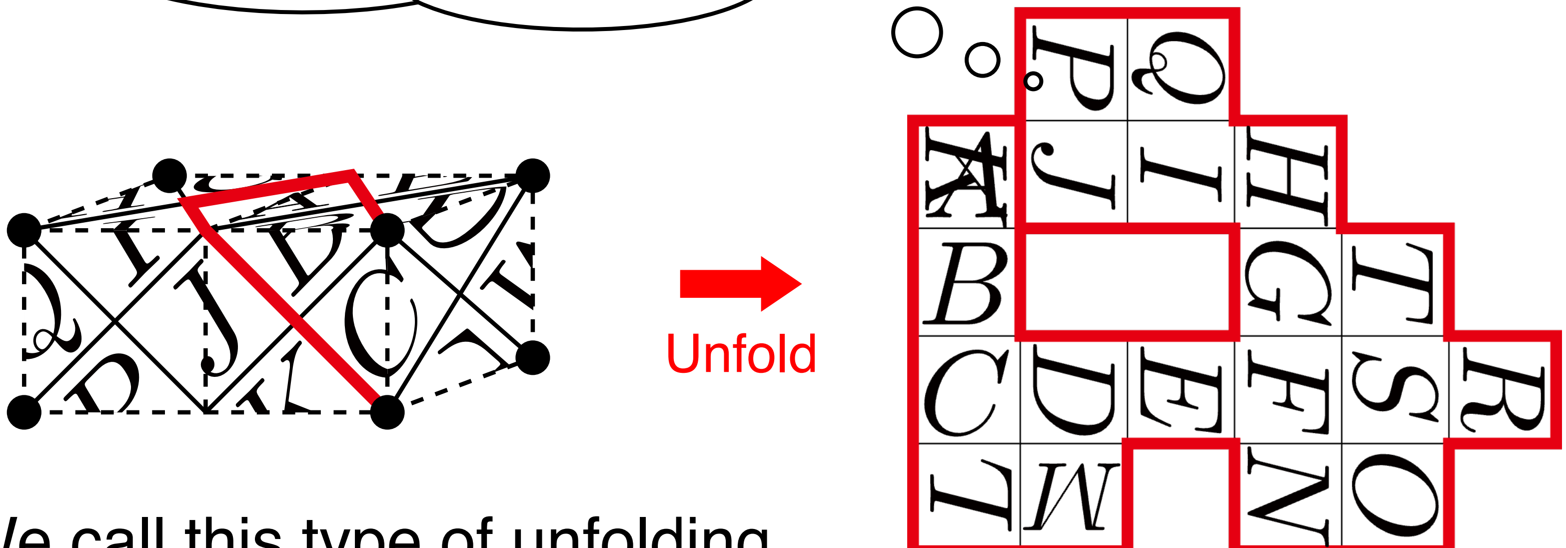
➢ We call this type of unfolding "Faces-in-touch unfolding".

Overlapping of lattice unfolding

Let's consider unfolding a cuboid into a polyomino.

[Note ... nnecting multiple

Hard to understand just
looking at this figure >:(



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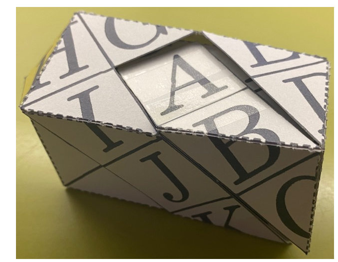
Mr. Kamata and I distribute my hand-made
3D models of "Faces-in-touch" for each table.



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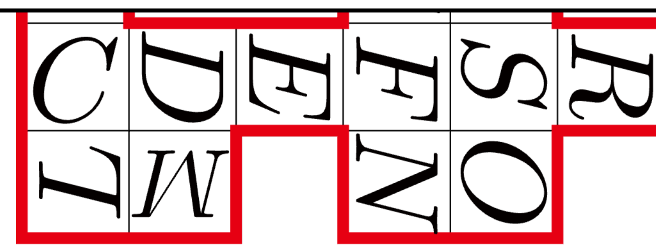

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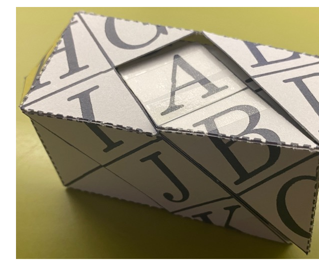
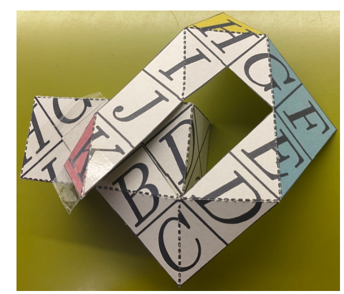


If you get the model …
Please look at it and unfold the model

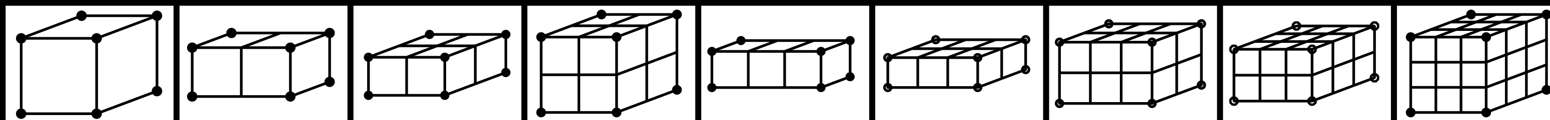
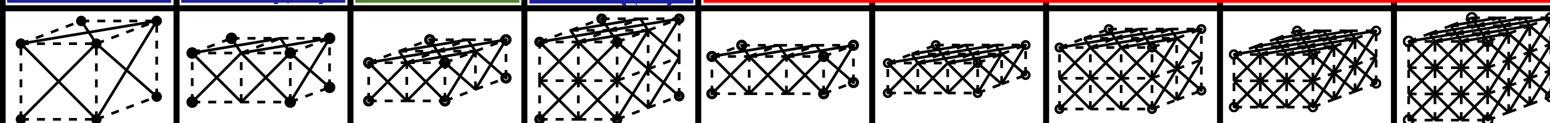
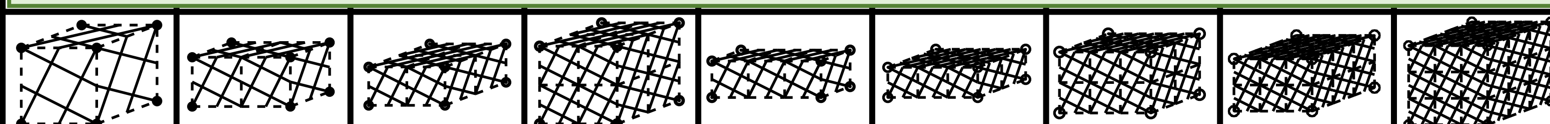
After you experience how they overlap …
Please fold the model again & pass it turn on the left/right



[†1](#) : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a,b) ※ $\gcd(a,b)=1$		(x,y,z)									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
	$(1,0)$										$\cdots$
	V / E / F	**No** (Obvi.)	**Open** (V, E) / **No (†1)** (F)	**Open**	**Open** (V, E) / **No (†2)** (F)	**Yes** $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	$(1,1)$										$\cdots$
	V / E / F	**Open**									
	$(2,1)$										$\cdots$
	V / E / F	**Open**									
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
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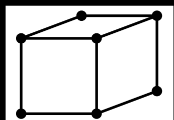
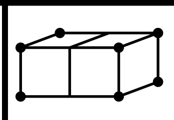
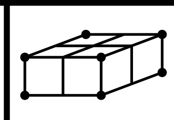
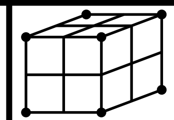
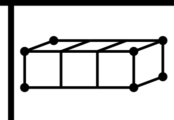
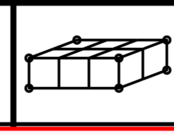
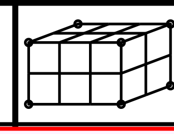
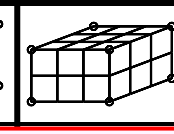
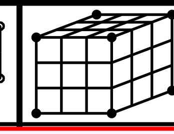
<tr><td colspan="2"></td><td colspan="10">(x, y, z)</td></tr>
<tr><td colspan="2"></td><td>$(1,1,1)$</td><td>$(1,1,2)$</td><td>$(1,2,2)$</td><td>$(2,2,2)$</td><td>$(1,1,3)$</td><td>$(1,2,3)$</td><td>$(2,2,3)$</td><td>$(2,3,3)$</td><td>$(3,3,3)$</td><td>$\cdots$</td></tr>
<tr><td rowspan="14">(a, b) ※ $\gcd(a,b) = 1$</td><td>$(1,0)$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\cdots$</td></tr>
<tr><td>V</td><td rowspan="3">No (Obvi.)</td><td>Yes</td><td colspan="2" rowspan="2">Yes</td><td colspan="6" rowspan="3">Yes $(1{\times}1{\times}z)$-cuboid: found by [T. Uno, 2008]
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<tr><td>E</td><td>No</td></tr>
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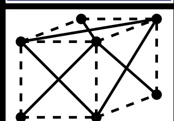
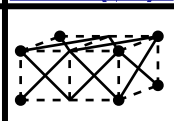
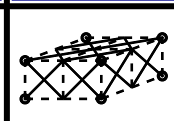
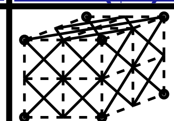
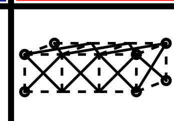
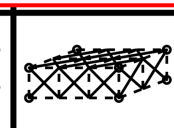
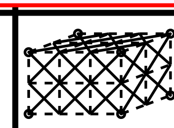
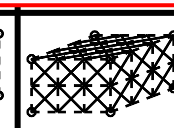
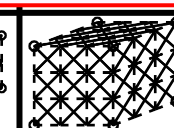
<tr><td>$(1,1)$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\cdots$</td></tr>
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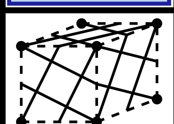
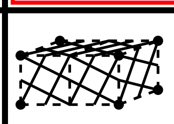
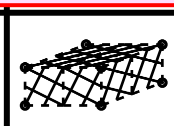
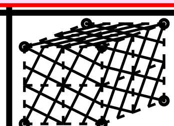
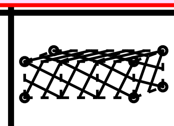
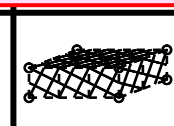
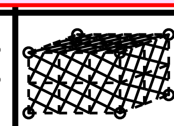
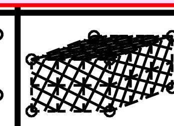
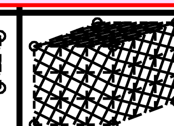
<tr><td>$(2,1)$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\cdots$</td></tr>
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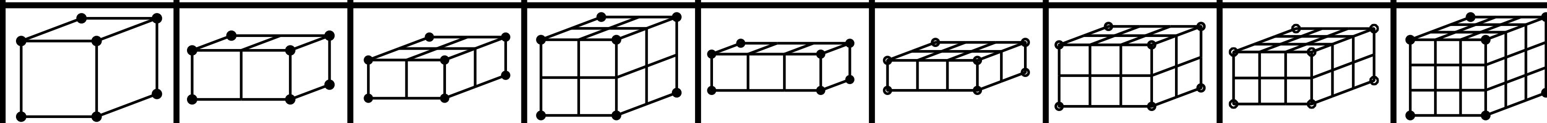
$(1,0)$											$\cdots$
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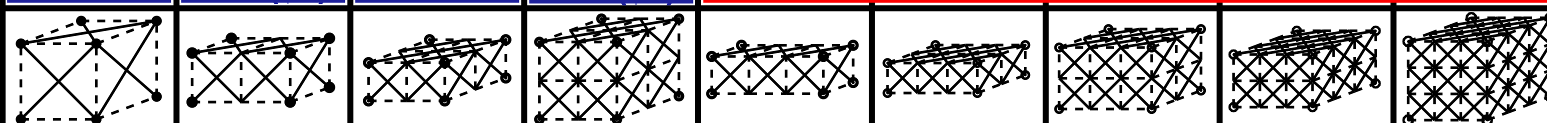
$(1,1)$											$\cdots$
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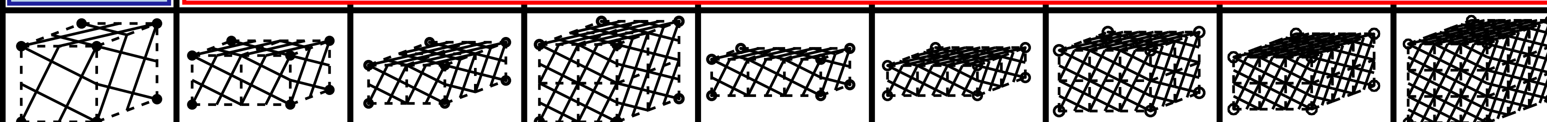
$(2,1)$											$\cdots$
	V										

To help understand how to read this table …

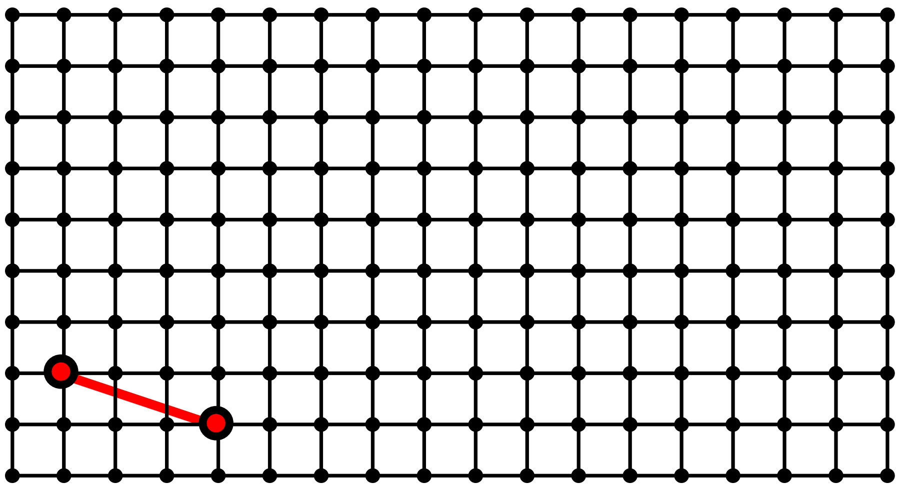
From the next slide, define the following three.

(1) Lattice cubes **(2)** Lattice cuboids **(3)** Lattice unfoldings

Lattice cubes

Definition 1

Choose two points on a square lattice and construct a square with these two points as one side. The cuboid assembled with this square as one face is called a lattice cube.



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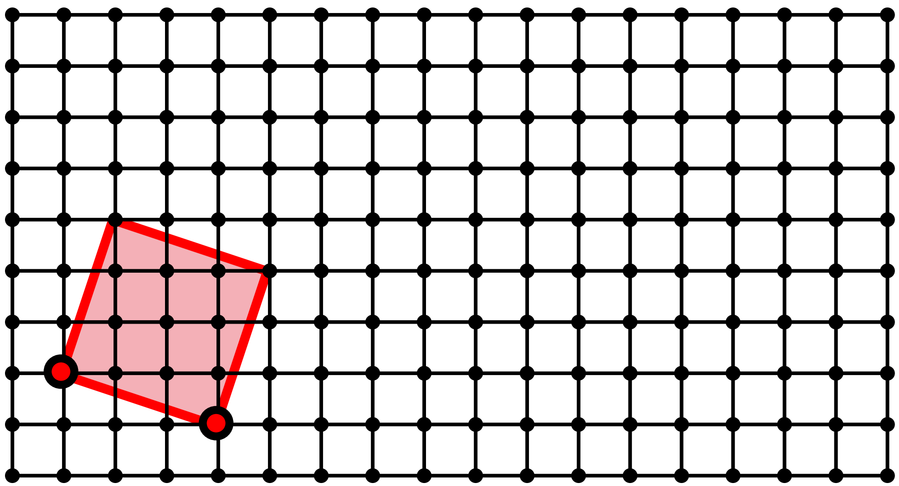

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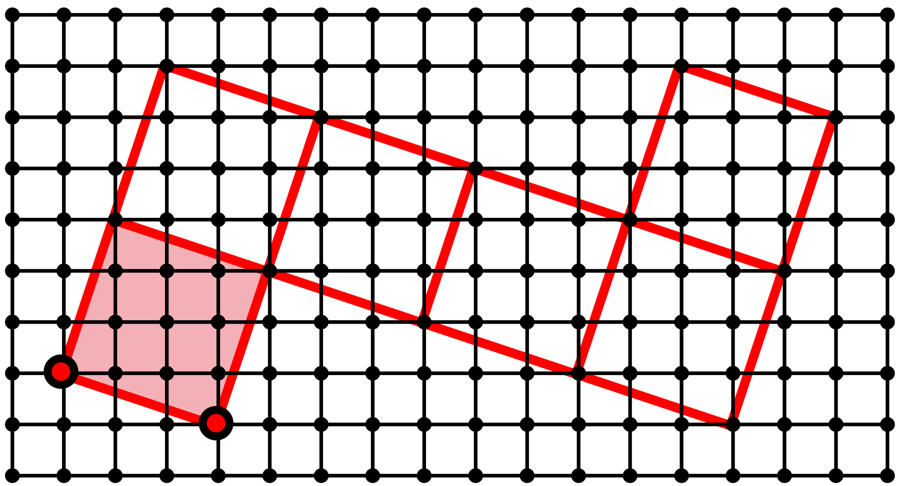

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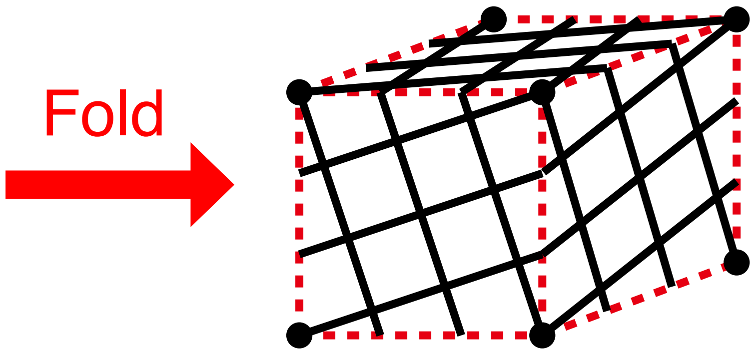


The lattice cube

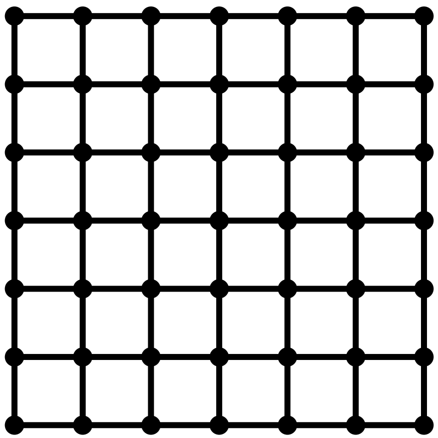
The length of one edge of a cube

We assume a square lattice of unit length (=1).

I. Choose a point $O(0,0)$ on the square lattice.

II. Let the coordinates of point A be $(a, 0)$ and B be $(0, b)$ $(\mathrm{a} \in \mathbb{N},\ \mathrm{b} \in \mathbb{N}^{+}, \mathrm{a} \geq \mathrm{b})$.

III. Let $L = |AB| = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$ be the length of one edge of a lattice cube.



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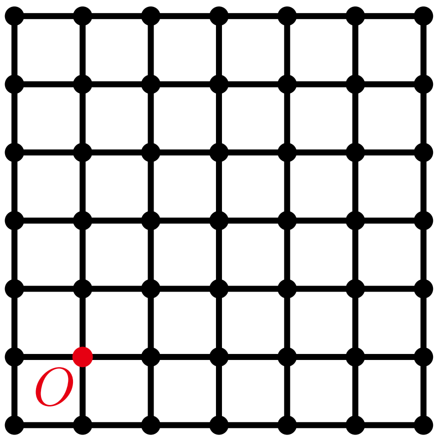

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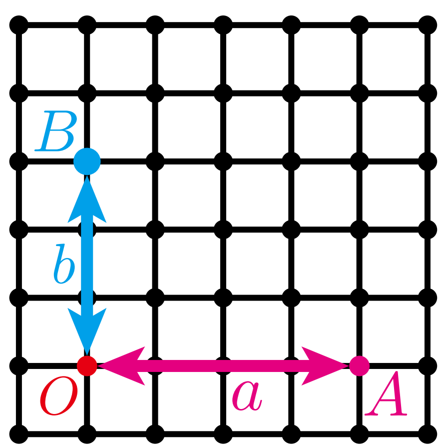

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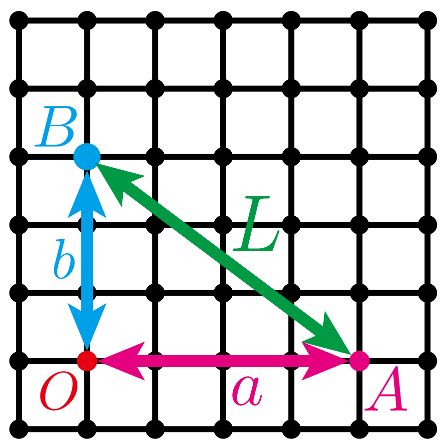
We assume a square lattice of unit length (=1).

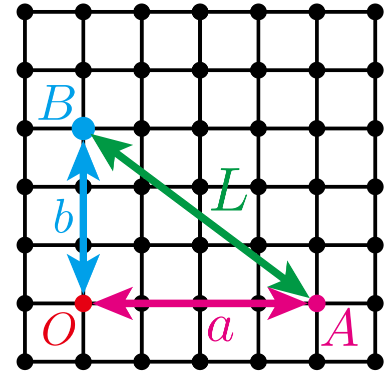
I. Choose a point $O(0,0)$ on the square lattice.

II. Let the coordinates of point A be $(a,0)$ and B be $(0,b)$ $(\mathrm{a} \in \mathbb{N},\ \mathrm{b} \in \mathbb{N}^{+}, \mathrm{a} \geq \mathrm{b})$.

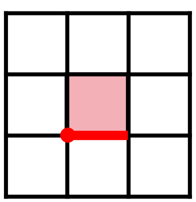
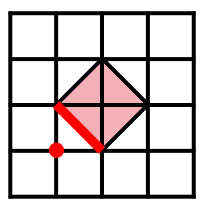
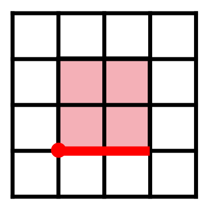
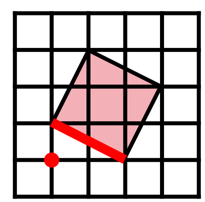
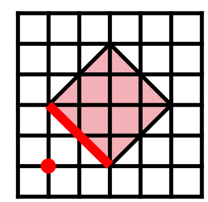
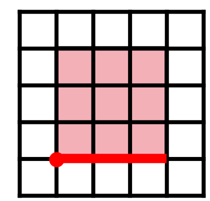
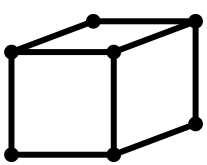
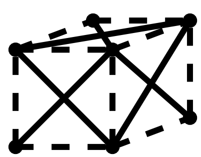
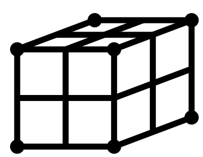
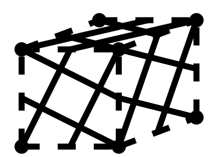
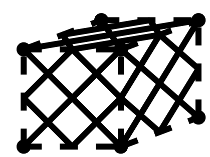
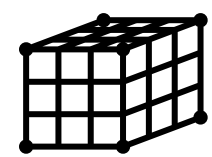
III. Let $L = |AB| = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$ be the length of one edge of a lattice cube.



The side length of a cube



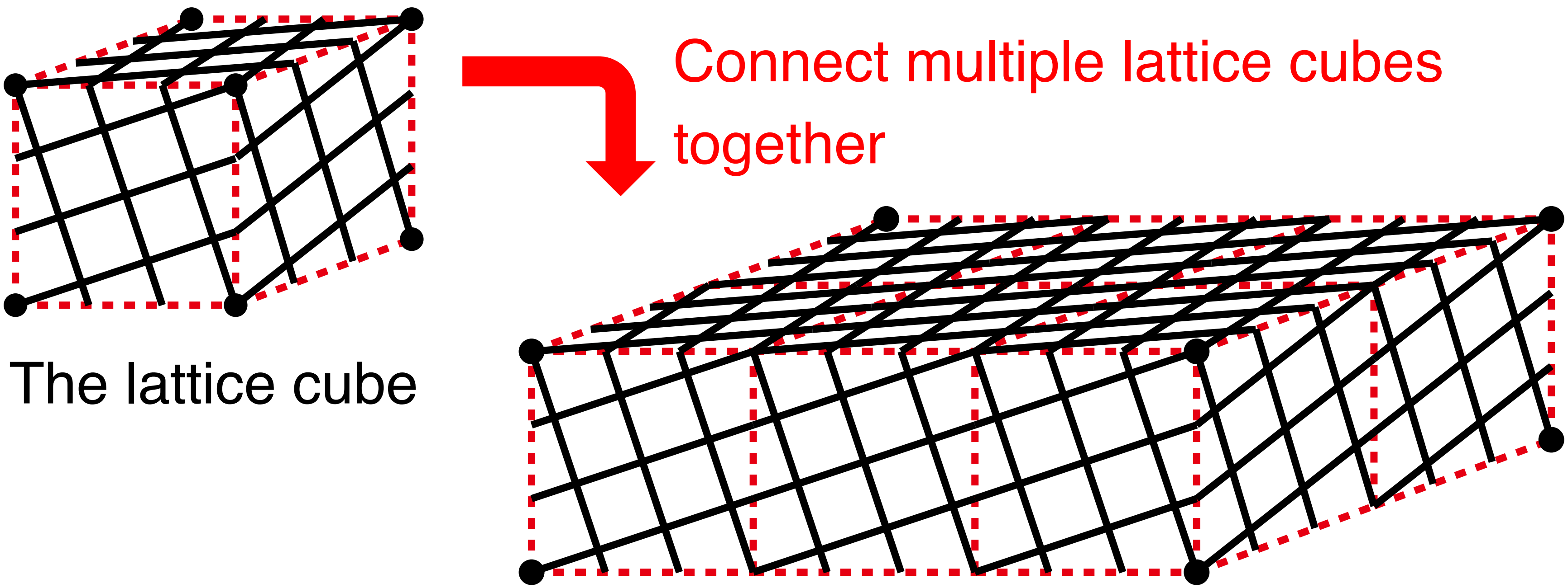
List of lattice cubes

a	1	1	2	2	2	3	⋯
b	0	1	0	1	2	0	⋯
L	1	$\sqrt{2}$	2	$\sqrt{5}$	$2\sqrt{2}$	3	⋯
$L\times L$ square							⋯
$L\times L\times L$ cube							⋯

Lattice cuboids

Definition 2

A cuboid made by connecting multiple lattice cubes is called a lattice cuboid. (Note: Lattice cubes $\subset$ Lattice cuboids)



The lattice cube

The lattice cuboid

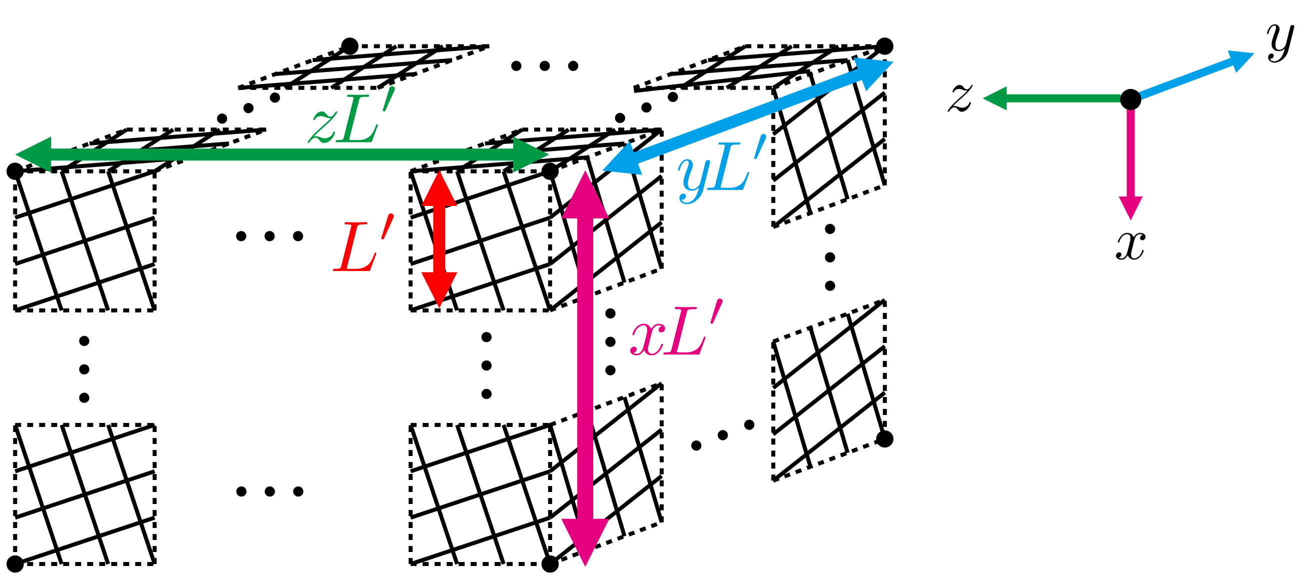
The three side lengths of a cuboid

Let L' be the length of one edge of a lattice cube.

$$L' = \sqrt{a^2 + b^2}\ (a \in \mathbb{N}^+, b \in \mathbb{N}, a \geq b, \gcd(a,b) = 1)$$

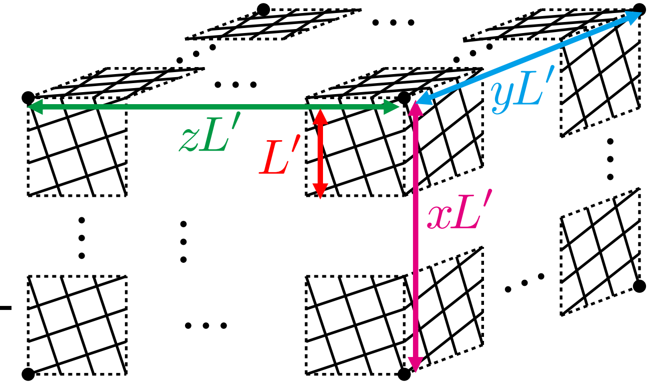
Denote the lattice cuboid as "(xL', yL', zL')-cuboid".

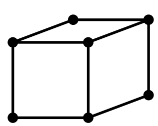
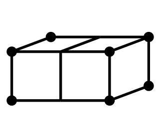
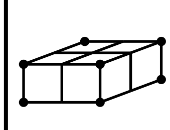
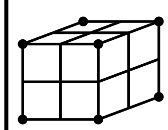
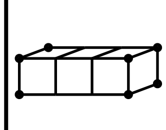
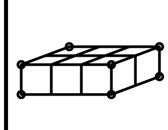
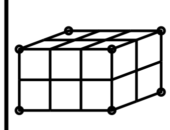
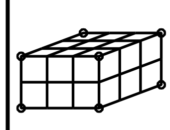
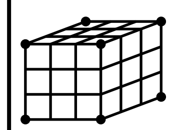
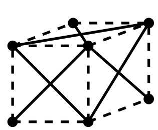
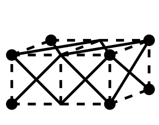
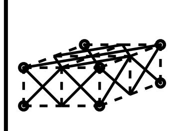
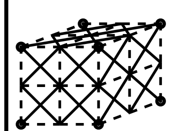
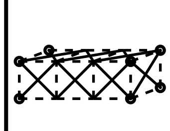
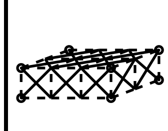
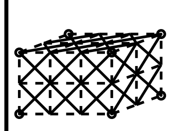
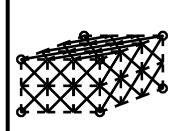
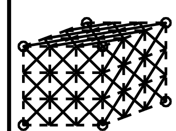
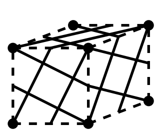
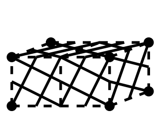
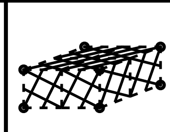
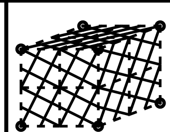
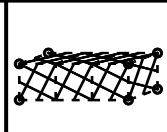
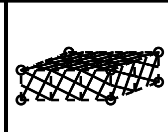
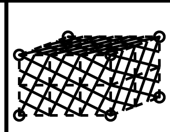
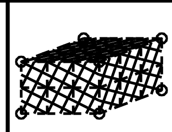
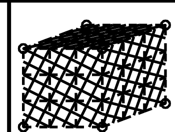
$$(x, y, z \in \mathbb{N},\ x \leq y \leq z)$$



The three side lengths of a cuboid

List of lattice cuboids

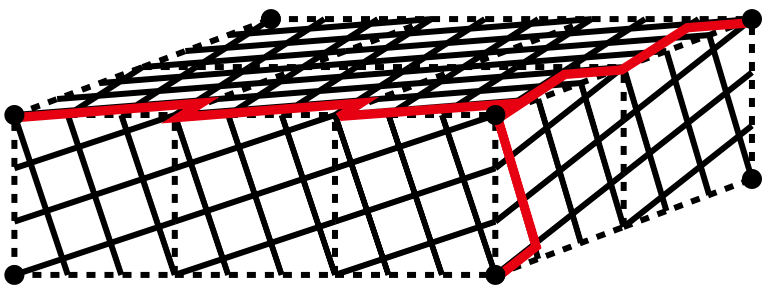


$(\boldsymbol{a},\boldsymbol{b})$ ※ $\gcd(a,b)=1$		$(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
	$(1,0)$										$\cdots$
	$(1,1)$										$\cdots$
	$(2,1)$										$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Lattice unfolding for cuboids

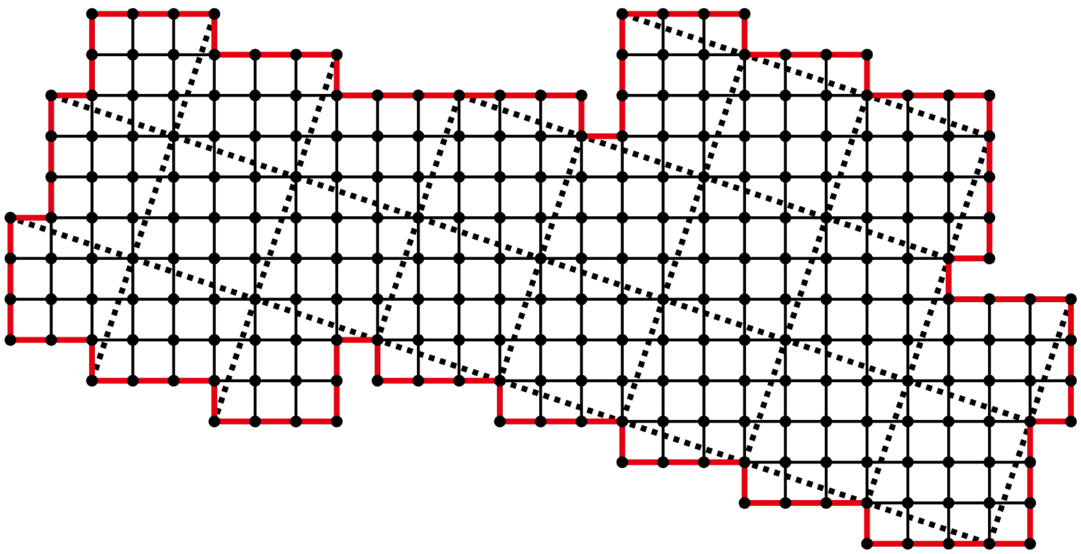
Definition 3

A lattice unfolding is a polygon obtained by cutting the face of the cuboid along the edges of unit squares.



The lattice cuboid

Cut along the edges of unit squares and unfold it flat.



Lattice unfolding for cuboids

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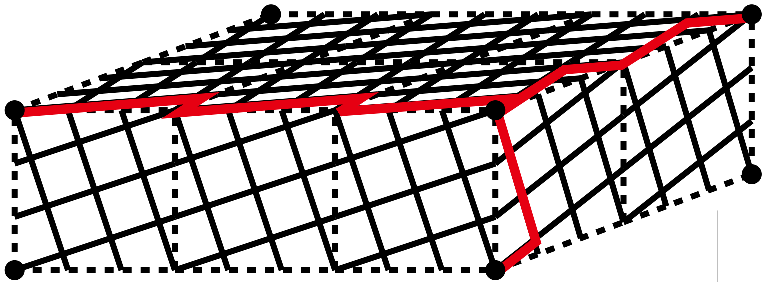
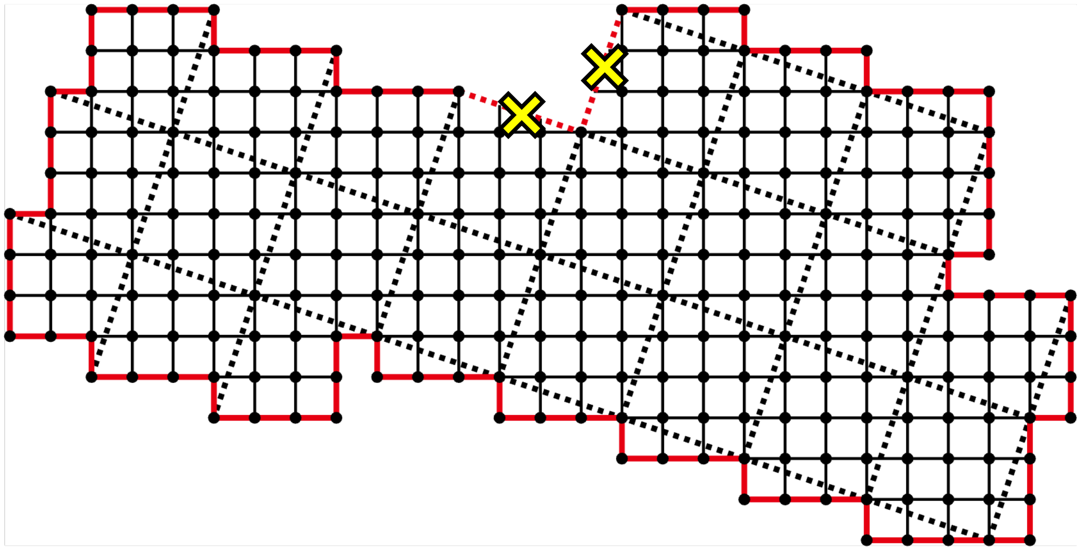


The lattice cuboid

Cut along the edges of unit squares and unfold it flat.



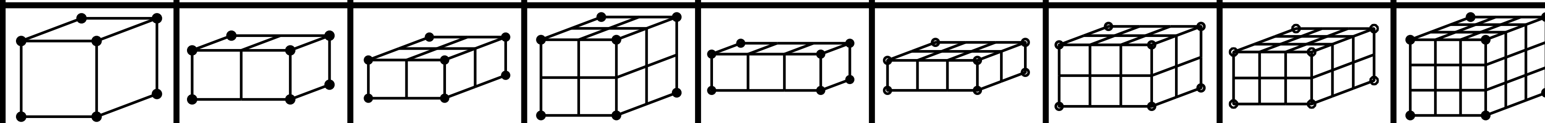
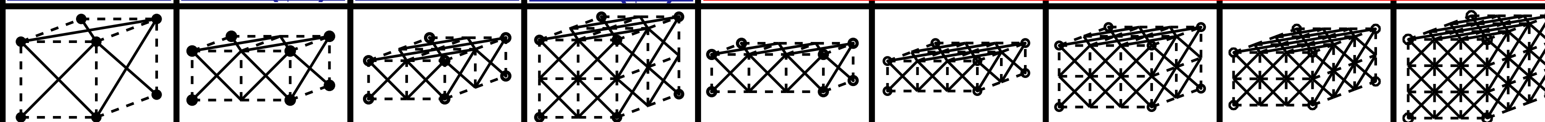
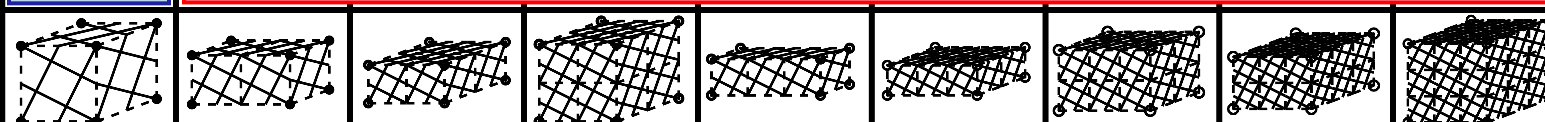
(Note)
Dotted lines ------ are folding lines (No cut)

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Background and our results

(a,b) ※ $\gcd(a,b)=1$

(a,b)		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
$(1,0)$											$\cdots$
	V	No (Obvi.)	Yes	Yes		Yes: $(1\times1\times z)$-cuboid: found by [T. Uno, 2008]; Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											$\cdots$
	V	No	Yes								
	E										
	F										
$(2,1)$											$\cdots$
	V	Yes									
	E										
	F										
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	V	Yes									
	E										
	F										

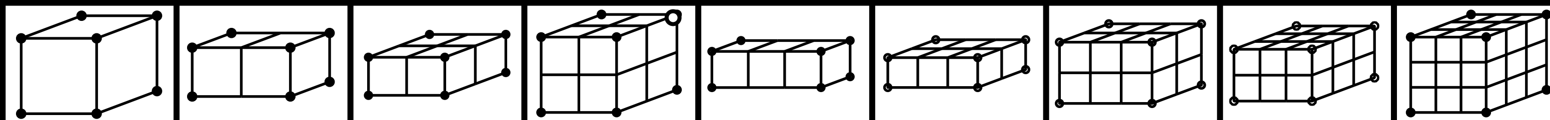
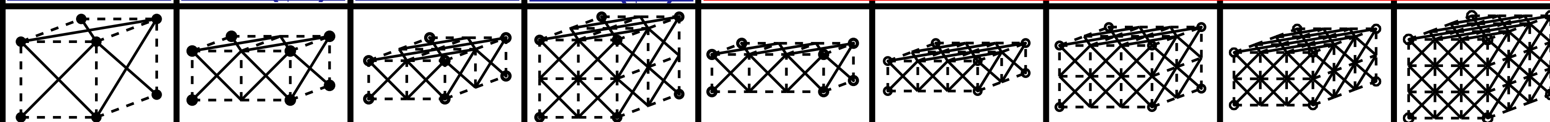
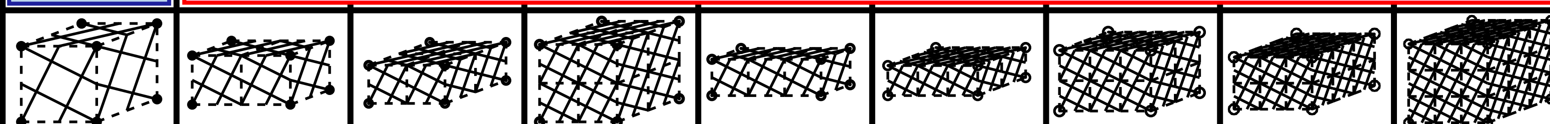
†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Re... ults

How do we prove them?

(a, b) ※ $\gcd(a, b) = 1$

(a,b)		(x,y,z): $(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
$(1,0)$											$\cdots$
	V	No (Obvi.)	Yes	Yes	Yes	Yes — $(1\times1\times z)$-cuboid: found by [T. Uno, 2008]; Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											$\cdots$
	V	No	Yes								
	E										
	F										
$(2,1)$											$\cdots$
	V	Yes									
	E										
	F										
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	V	Yes									
	E										
	F										

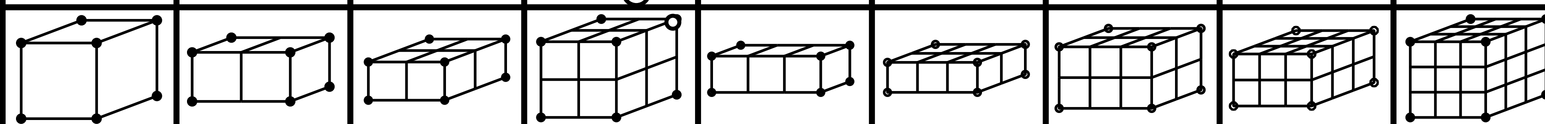
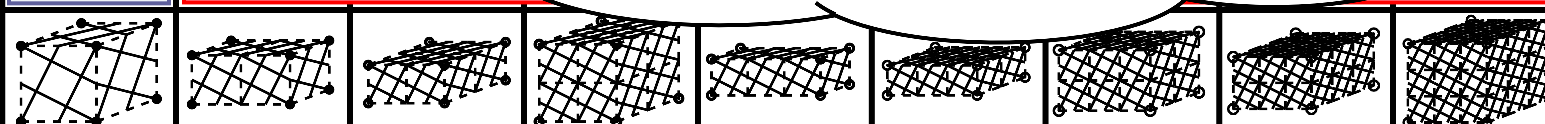
†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

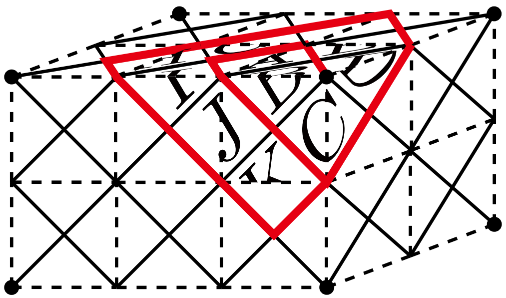
R… ults

How do we prove them?

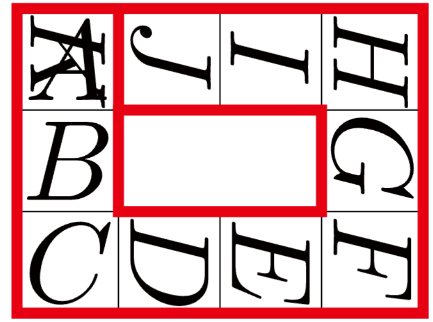
Demonstrate an example of overlapping lattice unfolding.

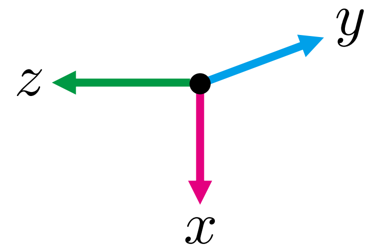
(a,b) ※ $\gcd(a,b)=1$		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes	Yes	Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											⋯
	V	No									
	E										
	F										
$(2,1)$											⋯
	V	Yes									
	E										
	F										
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

Technique to show the existence



Unfold

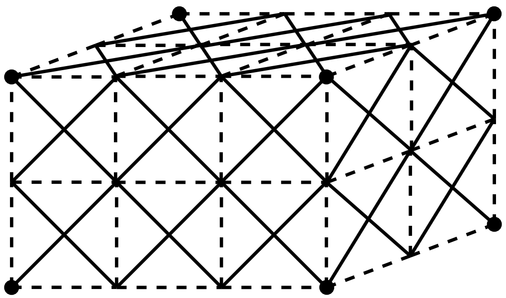




$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

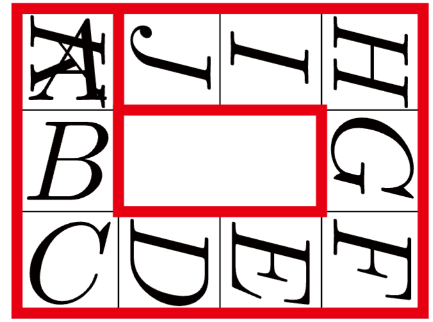
Lattice unfolding Q_1

Technique to show the existence

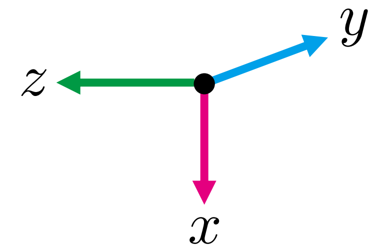


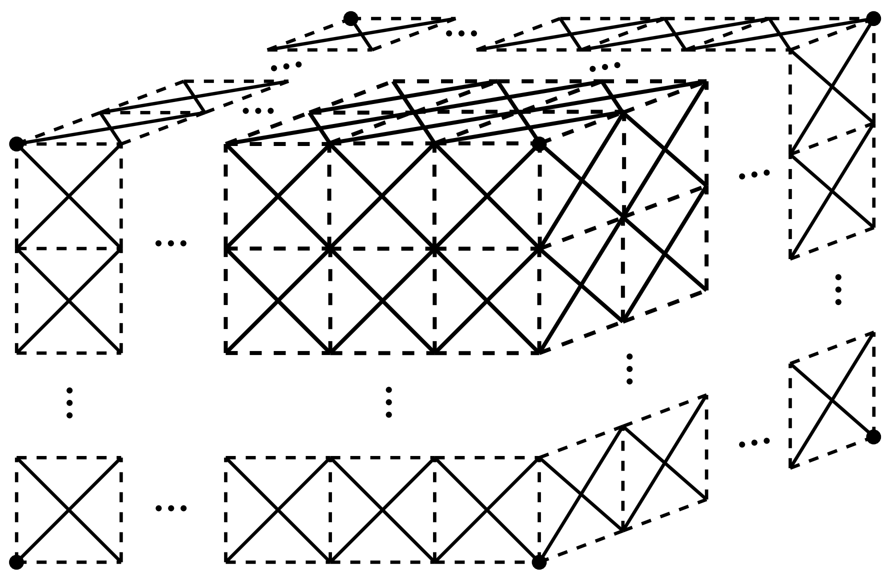
$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

Unfold



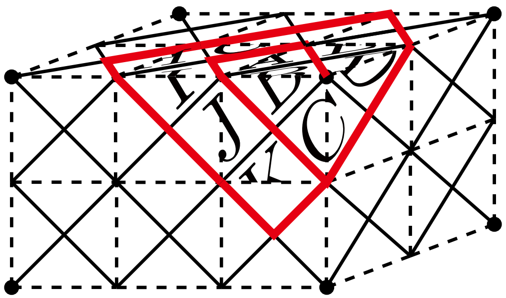
Lattice unfolding Q_1





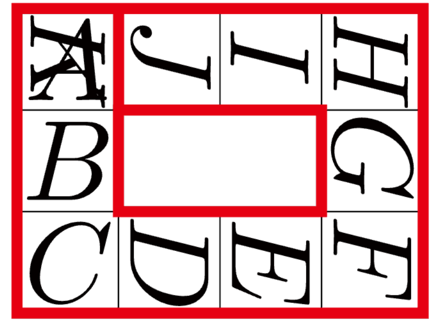
$(x\sqrt{2}, y\sqrt{2}, z\sqrt{2})$-cuboid $(x \geq 2, y \geq 2, z \geq 3)$

Technique to show the existence

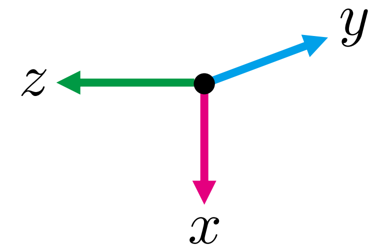


$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

Unfold



Lattice unfolding Q_1



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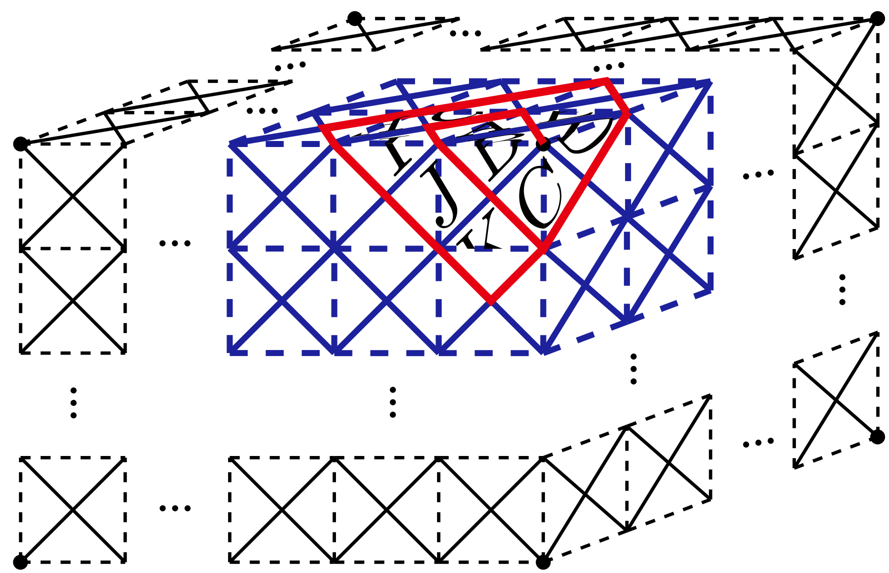
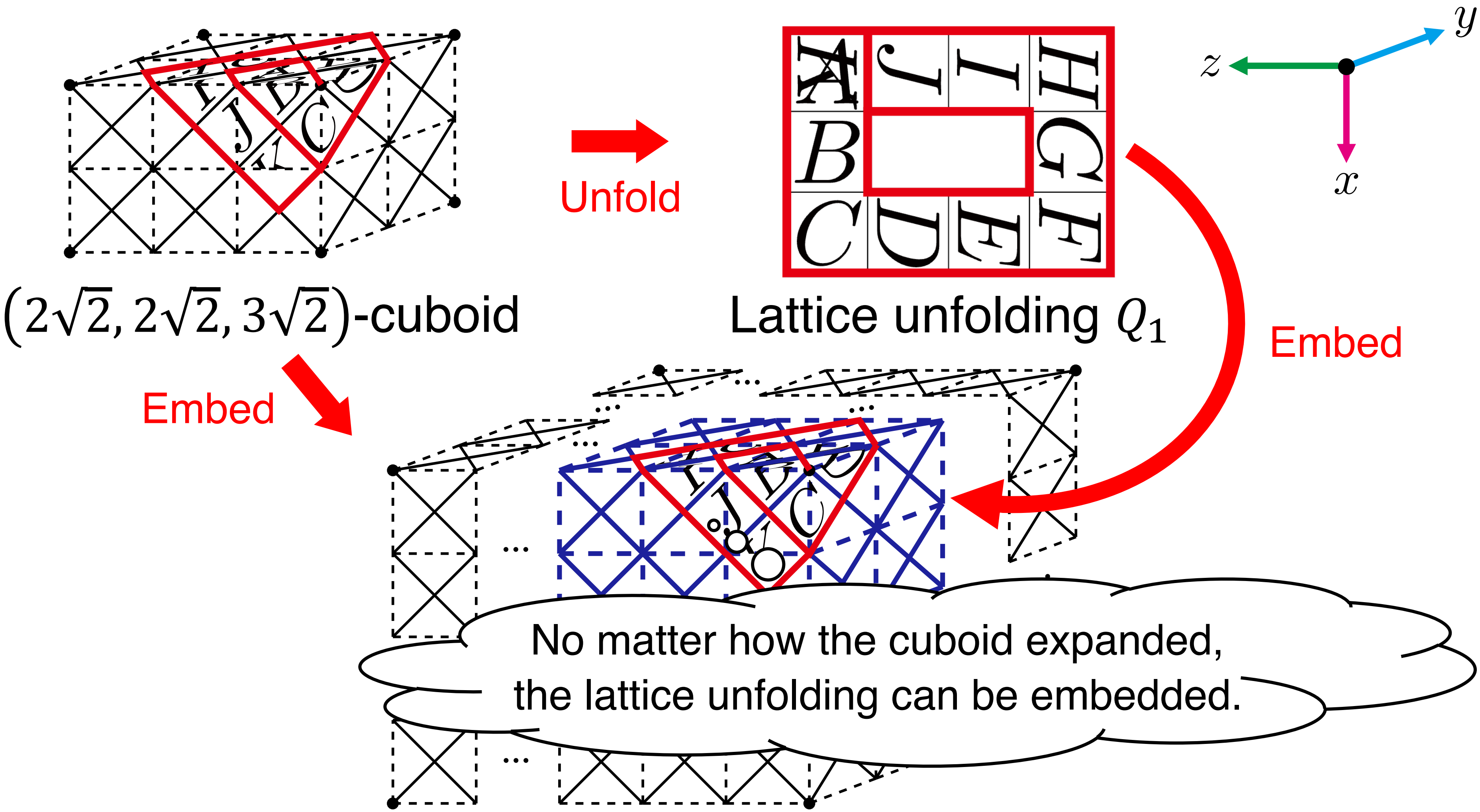


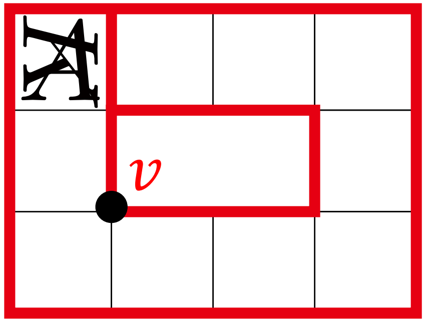
$(x\sqrt{2}, y\sqrt{2}, z\sqrt{2})$-cuboid $(x \geq 2, y \geq 2, z \geq 3)$

Technique to show the existence



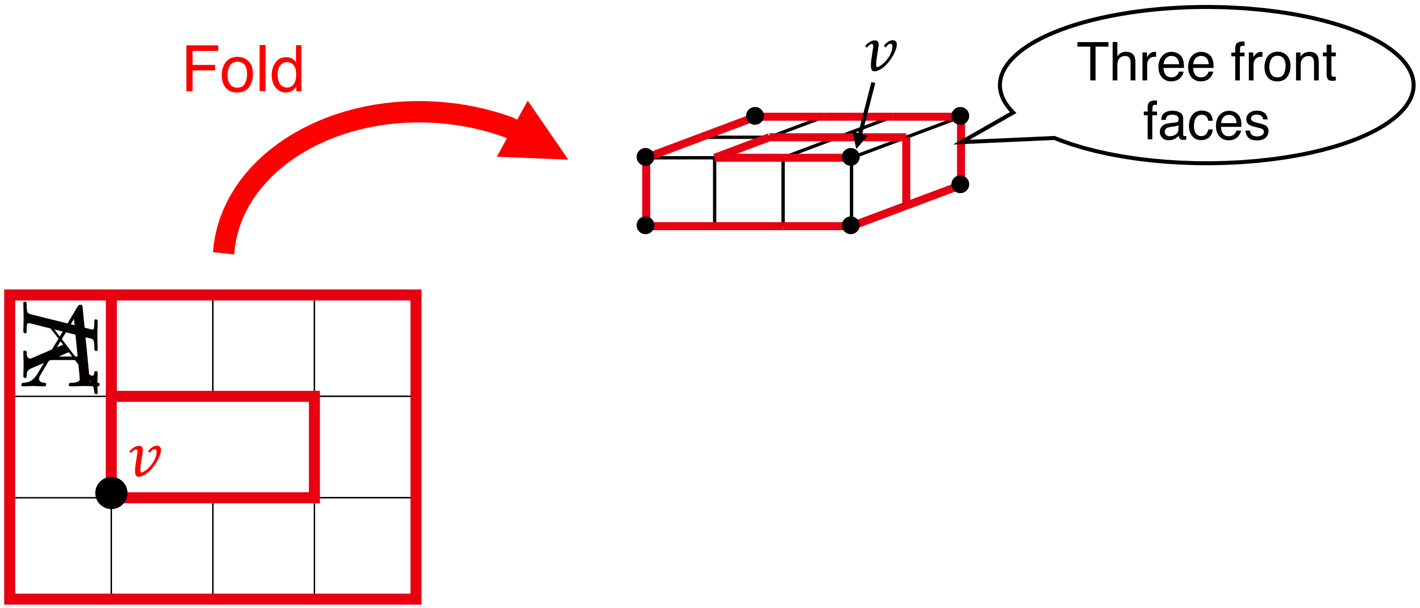
$(x\sqrt{2}, y\sqrt{2}, z\sqrt{2})$-cuboid $(x \geq 2, y \geq 2, z \geq 3)$

Technique to show the existence



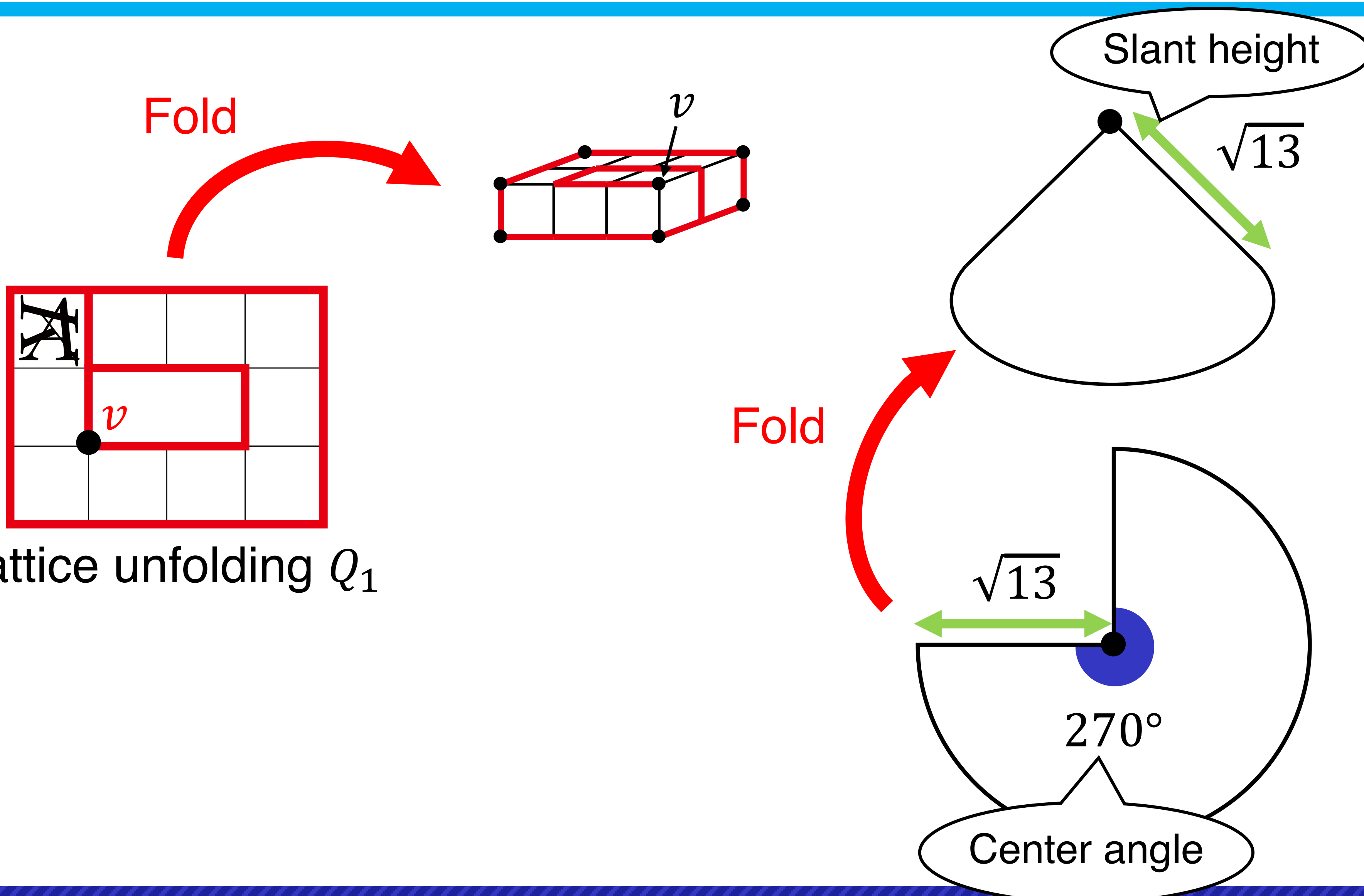
Lattice unfolding Q_1

Technique to show the existence



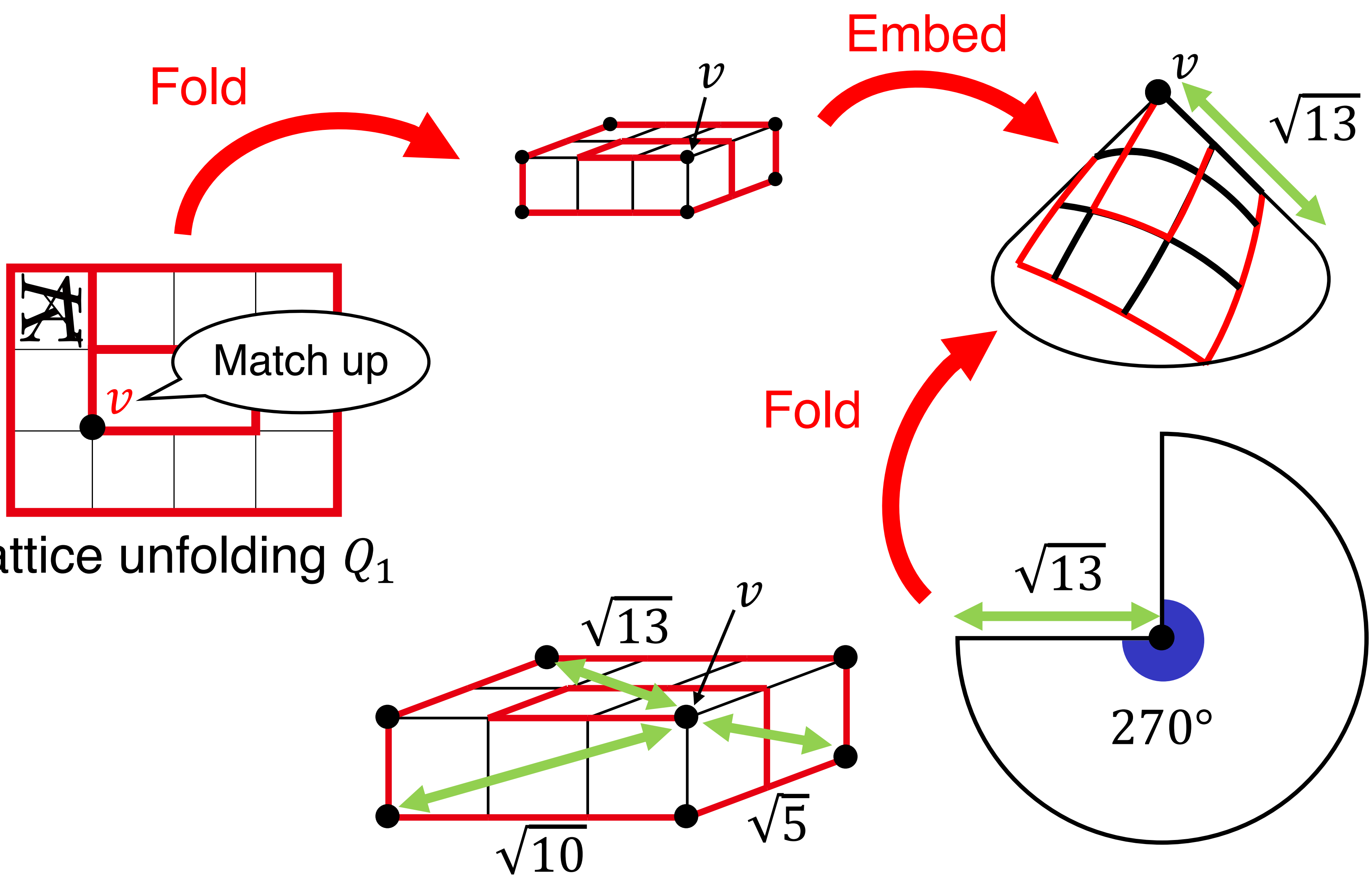
Lattice unfolding Q_1

Technique to show the existence



Lattice unfolding Q_1

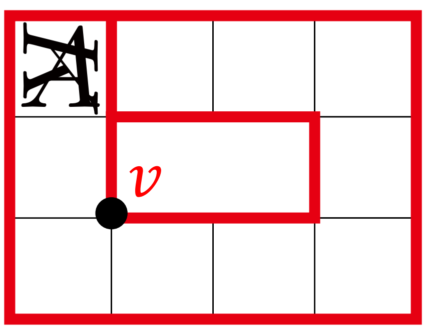
Technique to show the existence



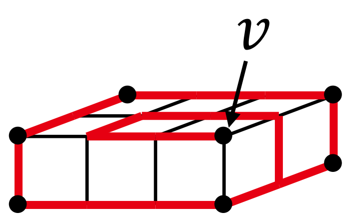
Lattice unfolding Q_1

Technique to show the existence

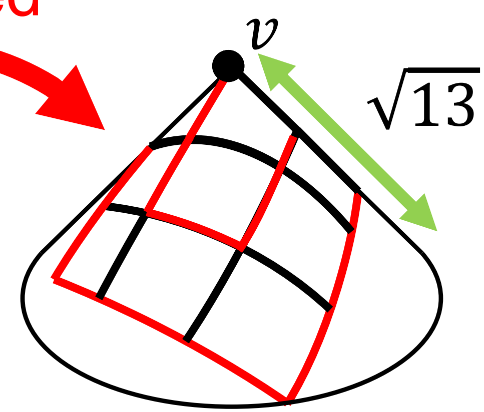
Fold



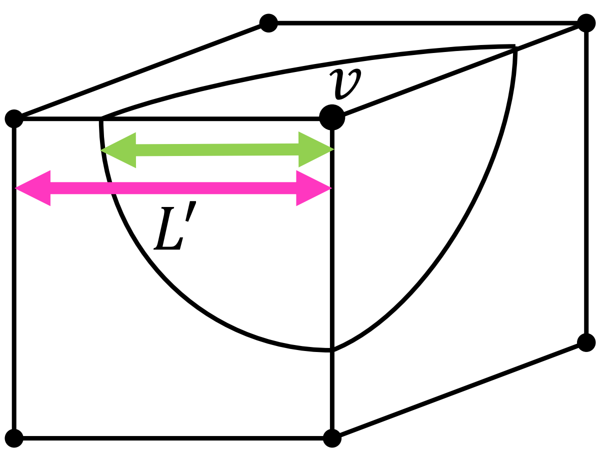
Lattice unfolding Q_1



Embed



Embed



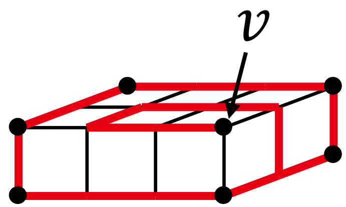
(L', L', L')-cuboid $(L' \geq \sqrt{13})$

[Note] $L' = \sqrt{a^2 + b^2}$

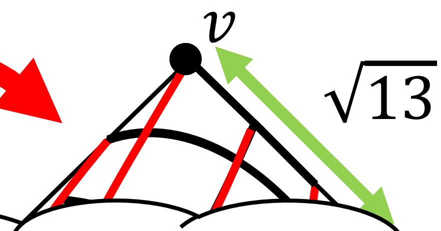
$(a \in \mathbb{N}^+, b \in \mathbb{N}, a \geq b)$

Technique to show the existence

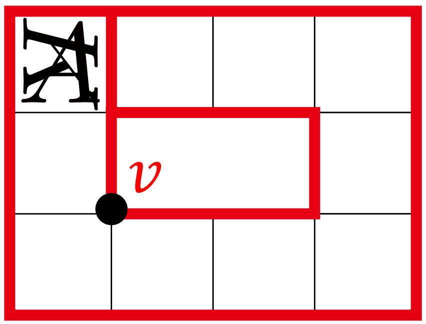
Fold



Embed



No matter what the size of L' ($\geq \sqrt{13}$), the lattice unfolding can be embedded.

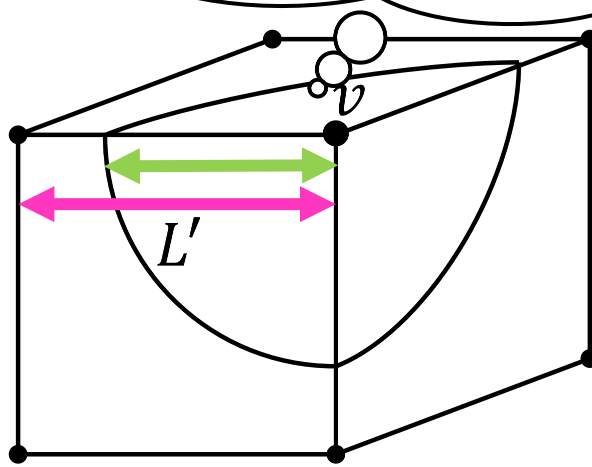


Lattice unfolding Q_1

Embed

[Note] $L' = \sqrt{a^2 + b^2}$

$(a \in \mathbb{N}^+, b \in \mathbb{N}, a \geq b)$

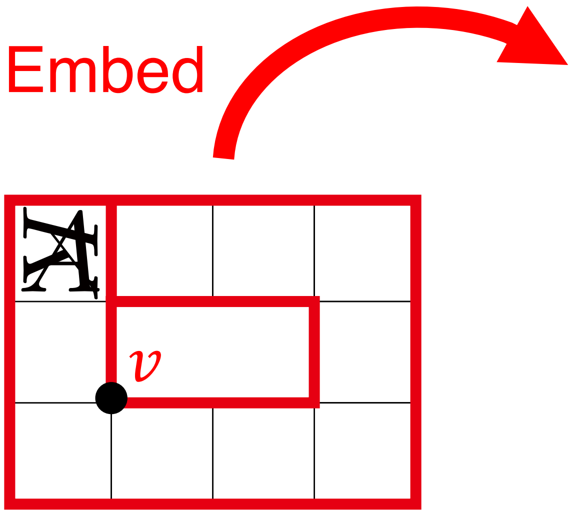


Embed

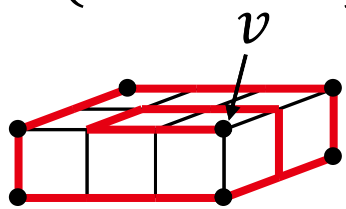
(L', L', L')-cuboid ($L' \geq \sqrt{13}$)

Technique to show the existence

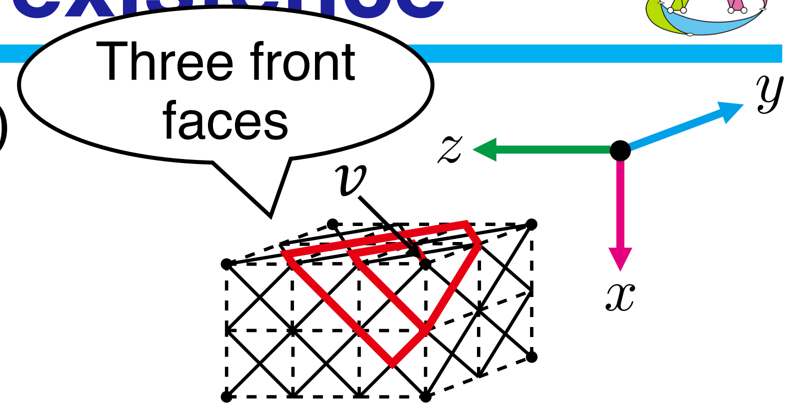
For (xL', yL', zL')-cuboid $(L' < \sqrt{13})$



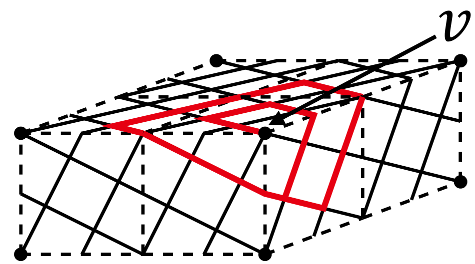
Lattice unfolding Q_1



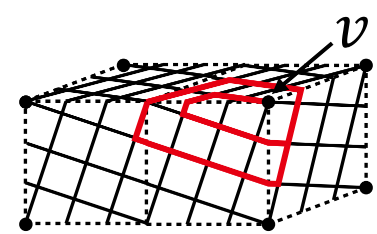
$(1,2,3)$-cuboid
[J. Mitani et al., 2008]



$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid



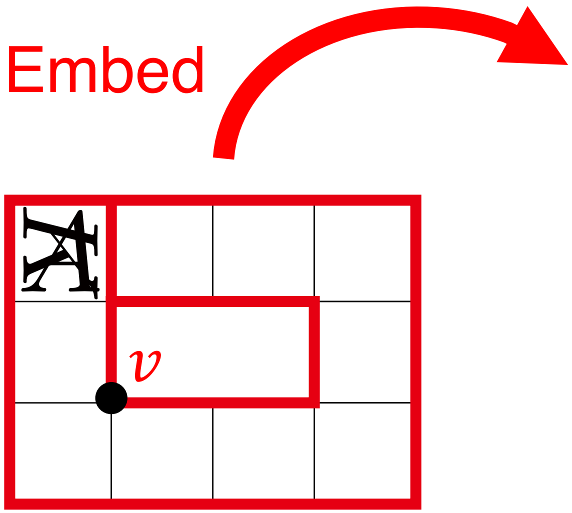
$(\sqrt{5}, 2\sqrt{5}, 2\sqrt{5})$-cuboid



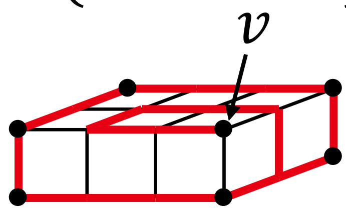
$(\sqrt{10}, \sqrt{10}, 2\sqrt{10})$-cuboid

Technique to show the existence

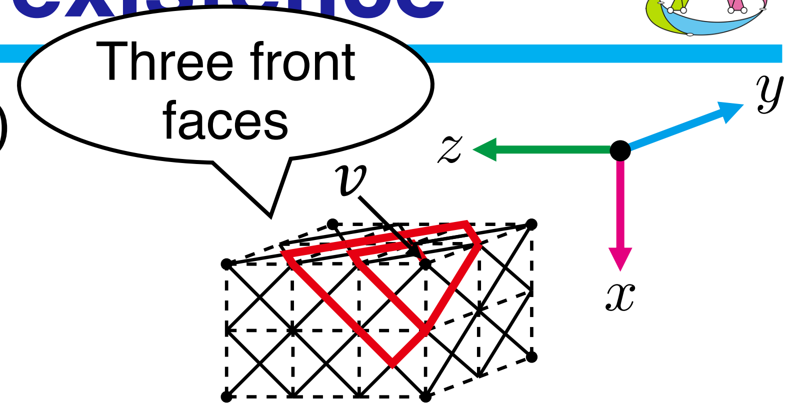
For (xL', yL', zL')-cuboid $(L' < \sqrt{13})$



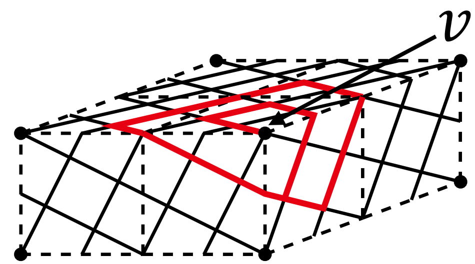
Lattice unfolding Q_1



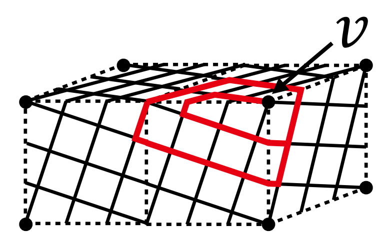
$(1,2,3)$-cuboid
[J. Mitani et al., 2008]



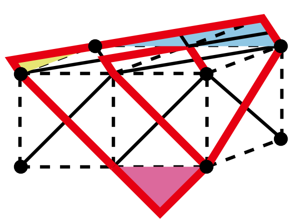
$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid



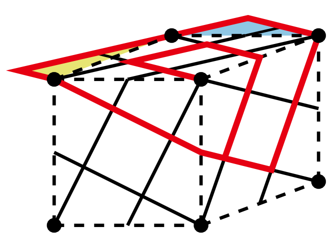
$(\sqrt{5}, 2\sqrt{5}, 2\sqrt{5})$-cuboid



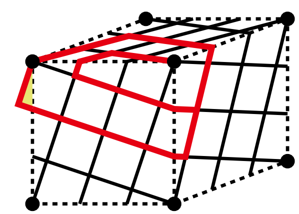
$(\sqrt{10}, \sqrt{10}, 2\sqrt{10})$-cuboid



$(\sqrt{2}, \sqrt{2}, 2\sqrt{2})$-cuboid



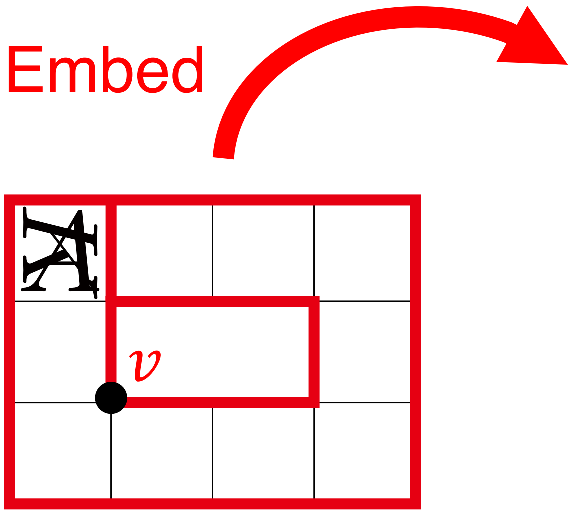
$(\sqrt{5}, \sqrt{5}, \sqrt{5})$-cuboid



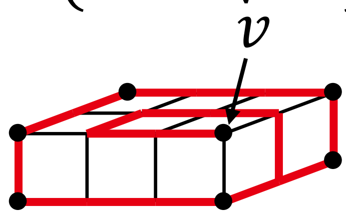
$(\sqrt{10}, \sqrt{10}, \sqrt{10})$-cuboid

Technique to show the existence

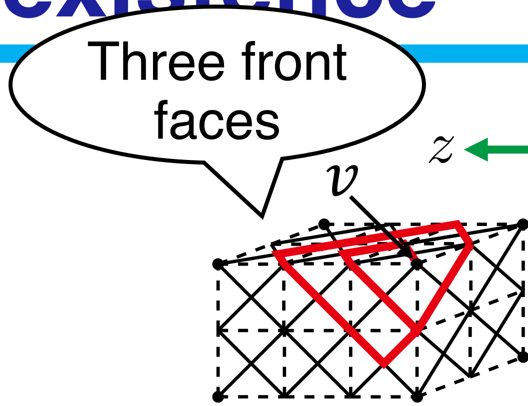
For (xL', yL', zL')-cuboid $(L' < \sqrt{13})$



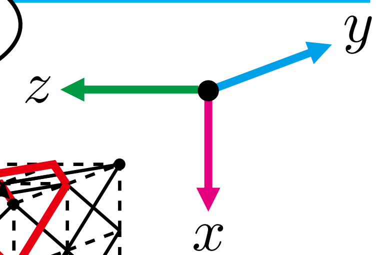
Lattice unfolding Q_1

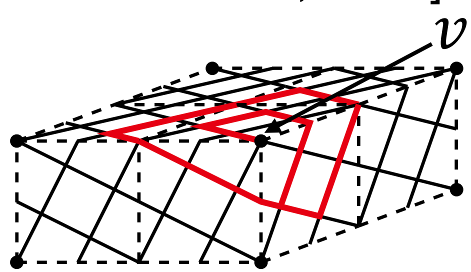


(1,2,3)-cuboid
[J. Mitani et al., 2008]

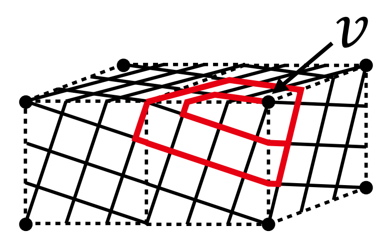


$(2\sqrt{2}, 2\sqrt{2}, 3\sqrt{2})$-cuboid

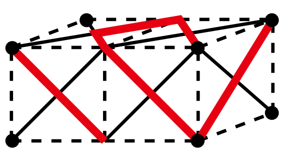




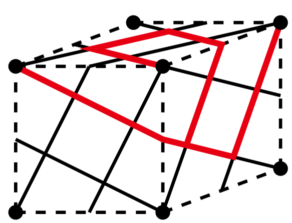
$(\sqrt{5}, 2\sqrt{5}, 2\sqrt{5})$-cuboid



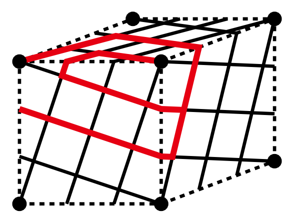
$(\sqrt{10}, \sqrt{10}, 2\sqrt{10})$-cuboid



$(\sqrt{2}, \sqrt{2}, 2\sqrt{2})$-cuboid



$(\sqrt{5}, \sqrt{5}, \sqrt{5})$-cuboid

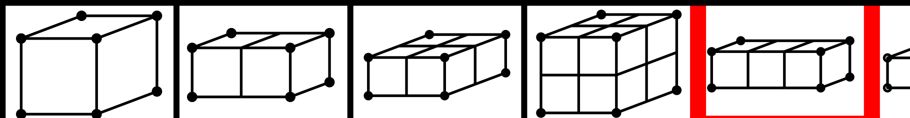
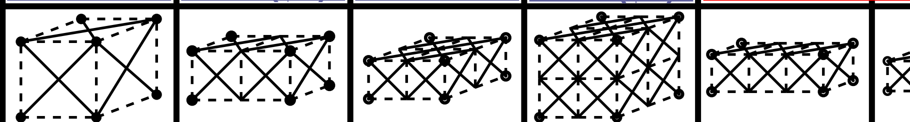
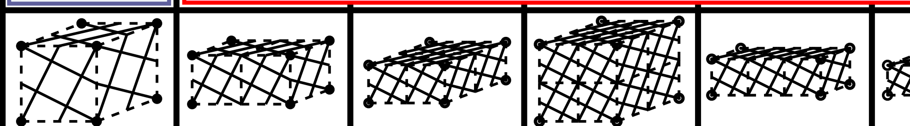


$(\sqrt{10}, \sqrt{10}, \sqrt{10})$-cuboid

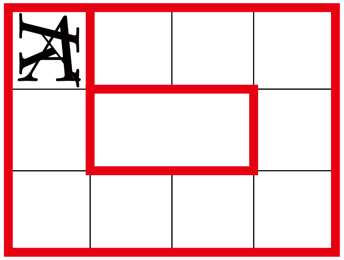
Background and our results

V	**Vertices-in-touch**
E	**Edges-in-touch**
F	**Faces-in-touch**

(a, b) ※ $\gcd(a, b) = 1$

(a,b)		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	(1
$(1,0)$							
	V	No (Obvi.)	Yes	Yes	Yes	Yes $(1\times$ Othe	
	E		No				
	F		No (†1)	No	No (†2)		
$(1,1)$							
	V	No	Ye				
	E						
	F						
$(2,1)$							
	V	Yes					
	E						
	F						
⋮		⋮	⋮	⋮	⋮	⋮	⋮
	V	Yes					
	E						
	F						

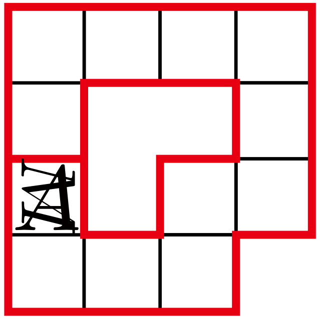
Gadgets for Faces-in-touch unfolding



Lattice unfolding Q_1

[Except]

$(1,1,z)$-cuboid $(z \geq 3)$

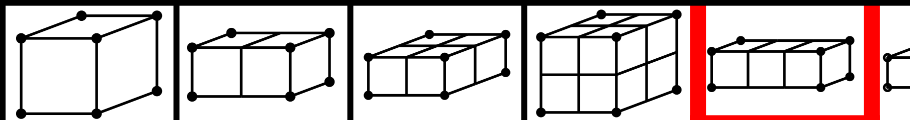
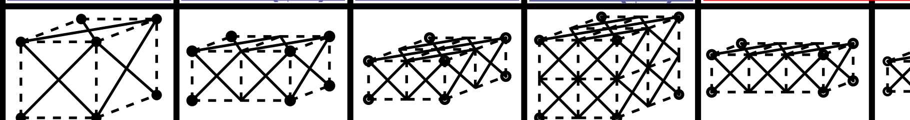
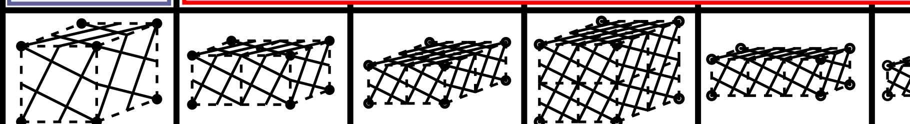


[T.Uno, 2008]

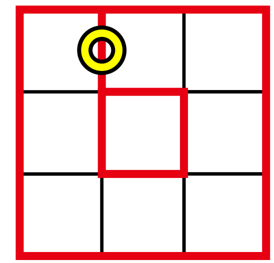
†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Background and our results

(a,b) ※ $\gcd(a,b)=1$		(x,y,z)					
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1$
$(1,0)$							
	V	No (Obvi.)	Yes	Yes		Yes $(1\times$ Othe	
	E		No				
	F		No (†1)	No	No (†2)		
$(1,1)$							
	V	No	Ye				
	E						
	F						
$(2,1)$							
	V	Yes					
	E						
	F						
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
	V	Yes					
	E						
	F						

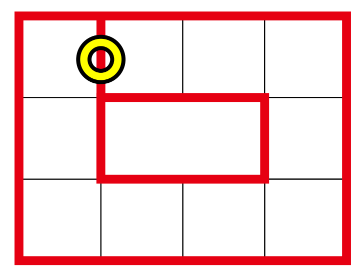
Gadgets for Edges-in-touch unfolding



Lattice unfolding Q_2

[Except]

$(1,1,z)$-cuboid $(z \geq 3)$



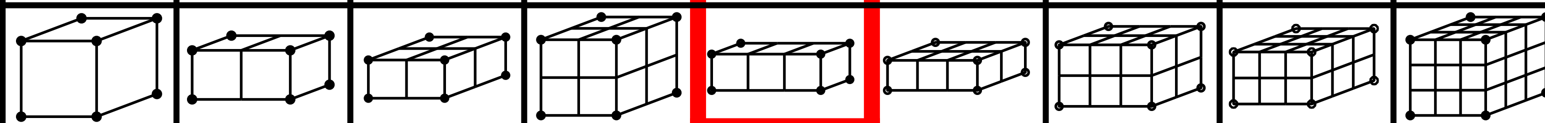
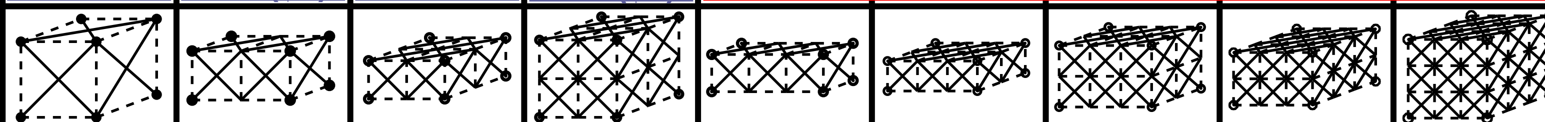
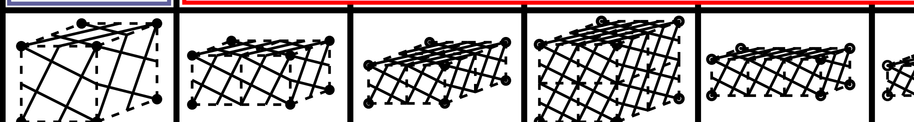
Lattice unfolding Q_1

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

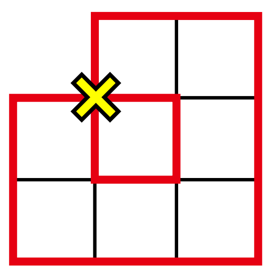
Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(x, y, z)

(a, b) ※ $\gcd(a,b)=1$		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes	Yes	Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											⋯
	V	No	Ye								
	E										
	F										
$(2,1)$											
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮				
	V	Yes									
	E										
	F										

Gadgets for Vertices-in-touch unfolding



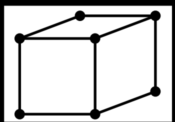
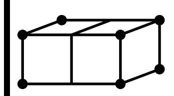
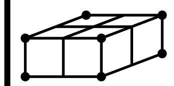
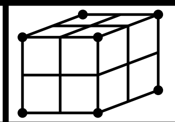
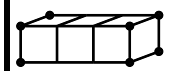
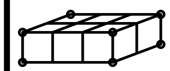
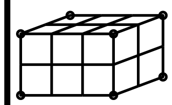
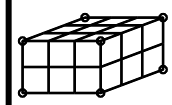
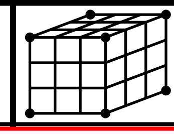
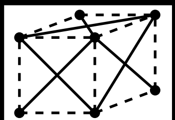
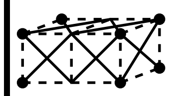
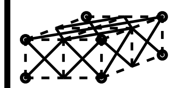
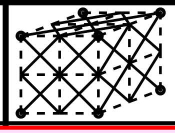
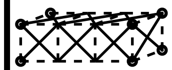
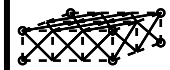
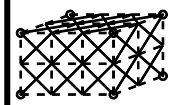
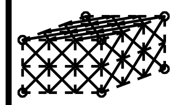
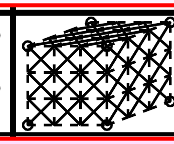
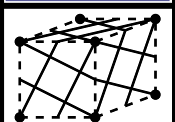
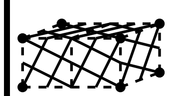
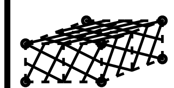
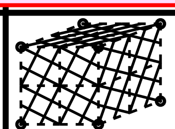
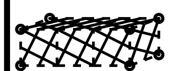
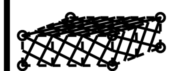
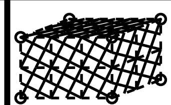
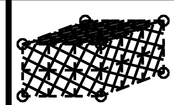
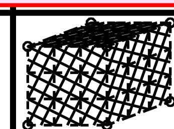
Lattice unfolding Q_3

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

Background and our results

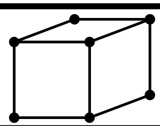
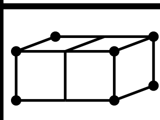
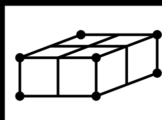
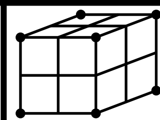
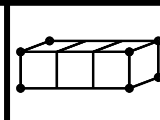
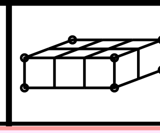
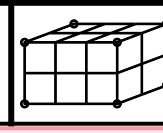
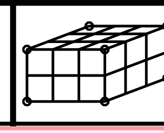
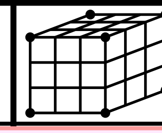
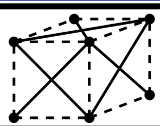
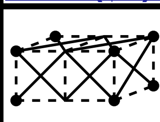
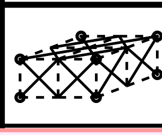
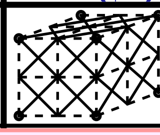
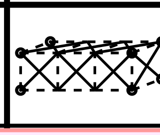
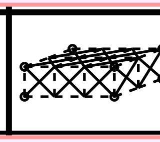
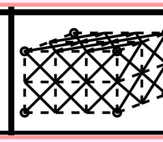
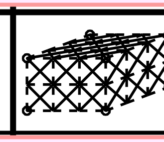
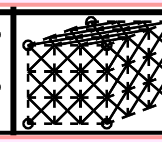
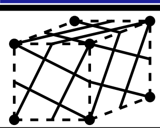
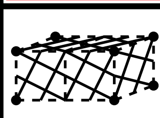
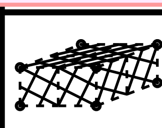
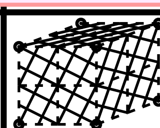
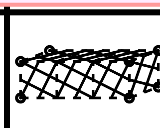
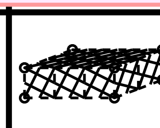
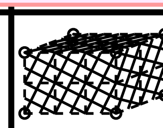
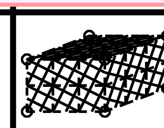
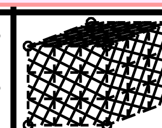
(a,b) ※ $\gcd(a,b)=1$

(a,b)		(x,y,z): $(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
$(1,0)$											$\cdots$
	V	No (Obvi.)	Yes	Yes	Yes	Yes: $(1\times1\times z)$-cuboid: found by [T. Uno, 2008]; Otherwise: found by [J. Mitani et al., 2008]					
	E	No (Obvi.)	No	Yes	Yes	Yes					
	F	No (Obvi.)	No (†1)	No	No (†2)	Yes					
$(1,1)$											$\cdots$
	V	No	Yes								
	E	No	Yes								
	F	No	Yes								
$(2,1)$											$\cdots$
	V	Yes									
	E	Yes									
	F	Yes									
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	V	Yes									
	E	Yes									
	F	Yes									

Background and our results

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

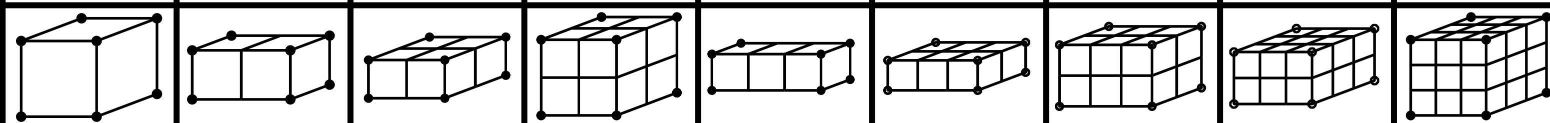
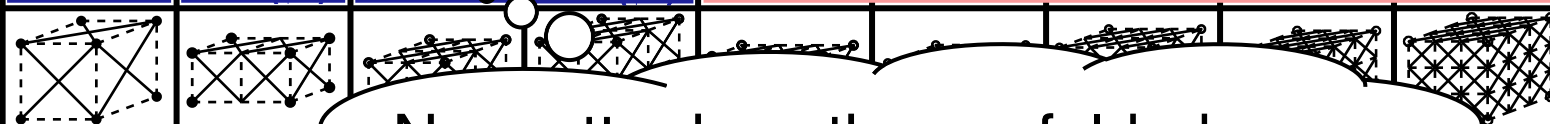
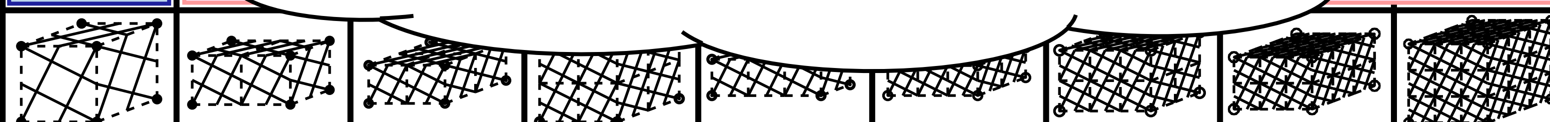
		(x,y,z)									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
(a,b) ※ $\gcd(a,b)=1$	$(1,0)$										$\cdots$
	V	No (Obvi.)	Yes	Yes		Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
	$(1,1)$										$\cdots$
	V	No	Yes								
	E										
	F										
	$(2,1)$										$\cdots$
	V	Yes									
	E										
	F										
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	V	Yes									
	E										
	F										

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a, b) ※ $\gcd(a, b) = 1$

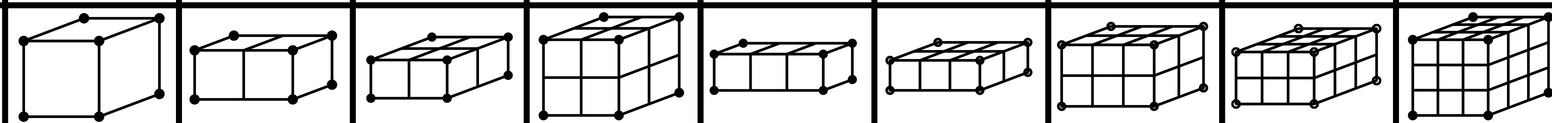
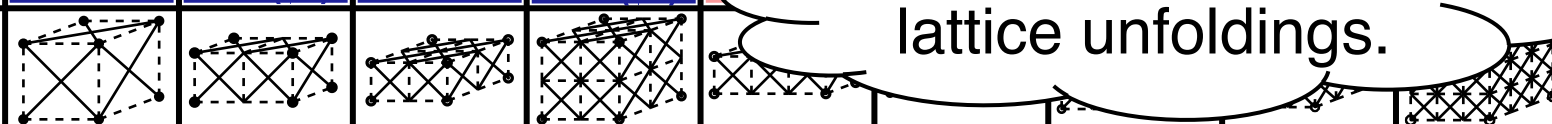
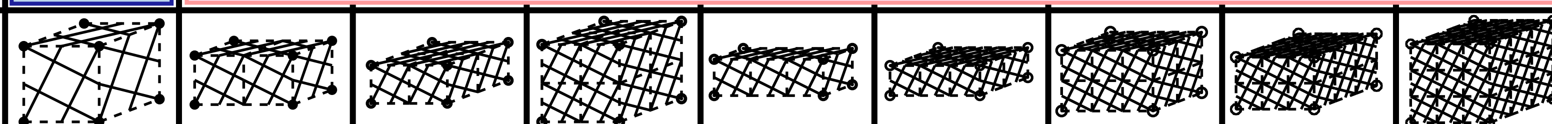
(a,b)		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	⋯
$(1,0)$											⋯
	V	No (Obvi.)	Yes	Yes	Yes	Yes $(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											⋯
	V	No									
	E										
	F										
$(2,1)$											⋯
	V	Yes									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	Yes									
	E										
	F										

No matter how they unfolded,
they do not overlap.

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

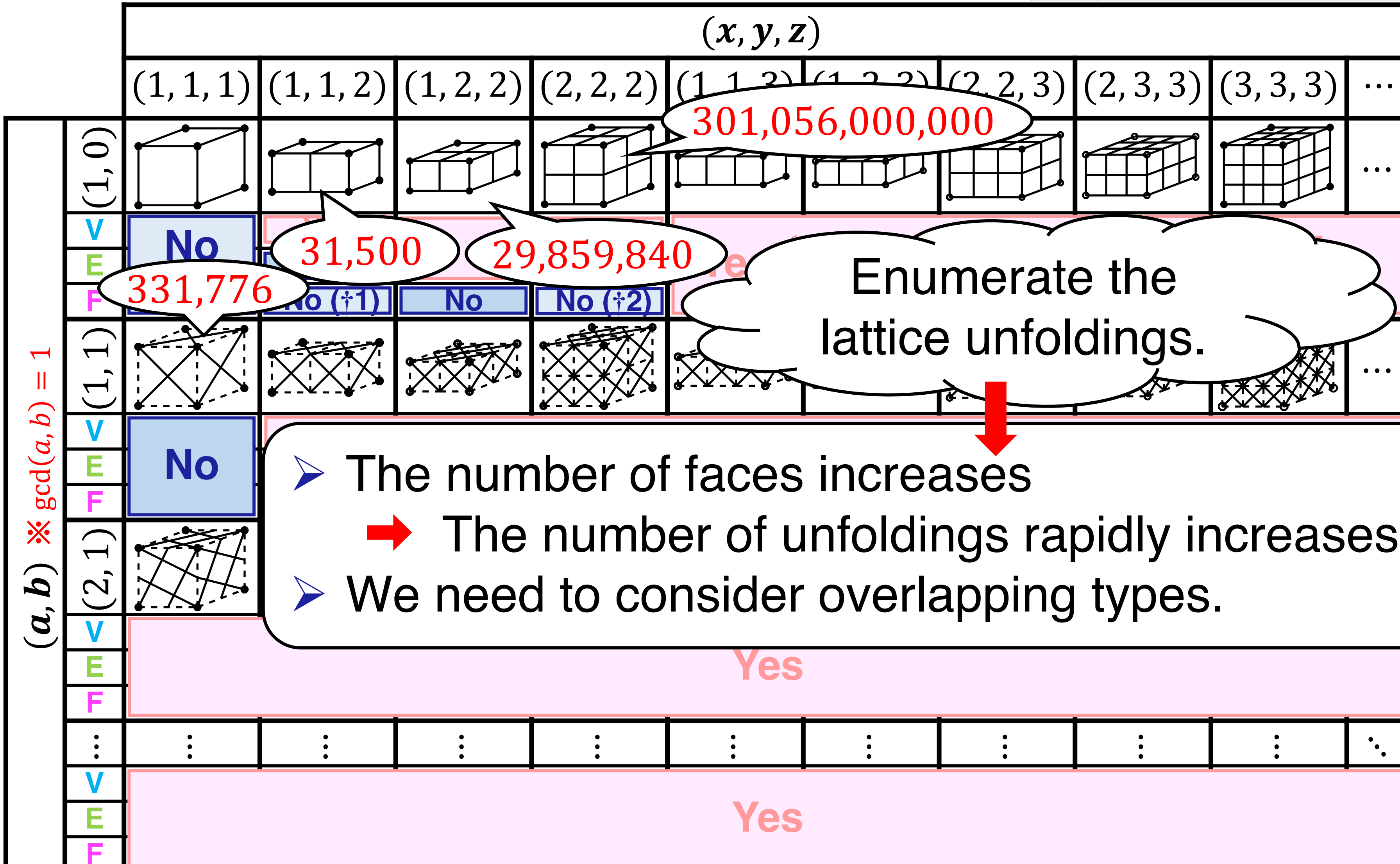
(a,b) ※ $\gcd(a,b)=1$		(x,y,z)									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
$(1,0)$											$\cdots$
	V	No (Obvi.)	Yes	Yes		Ye					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											$\cdots$
	V	No	Yes								
	E										
	F										
$(2,1)$											$\cdots$
	V	Yes									
	E										
	F										
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	V	Yes									
	E										
	F										

Enumerate the lattice unfoldings.

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

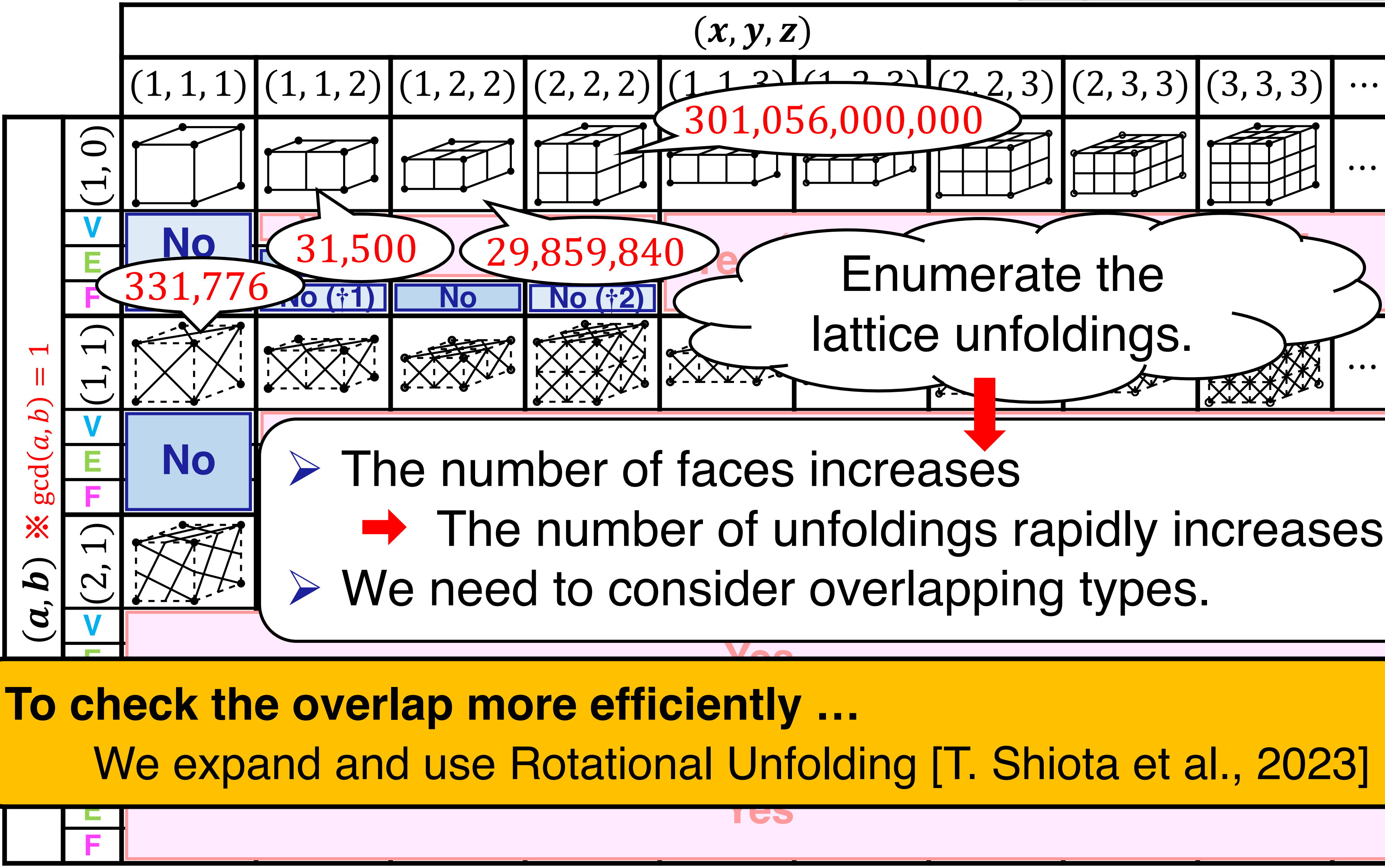


- The number of faces increases
  - ➡ The number of unfoldings rapidly increases
- We need to consider overlapping types.

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch



- The number of faces increases
  ➡ The number of unfoldings rapidly increases
- We need to consider overlapping types.

To check the overlap more efficiently …

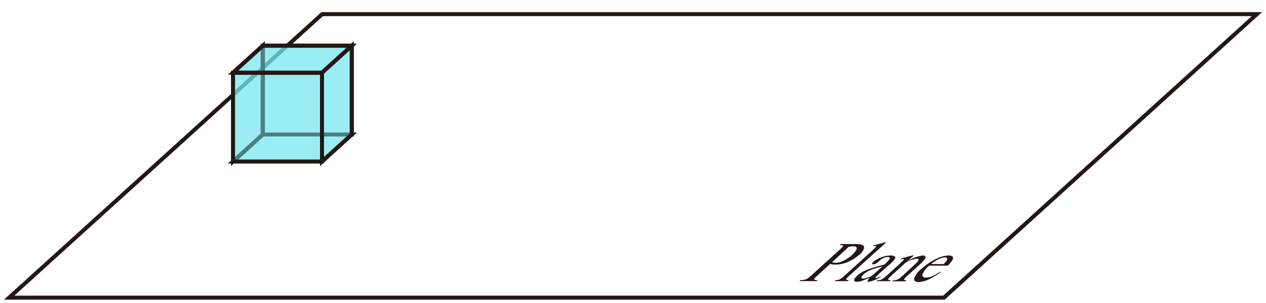
We expand and use Rotational Unfolding [T. Shiota et al., 2023]

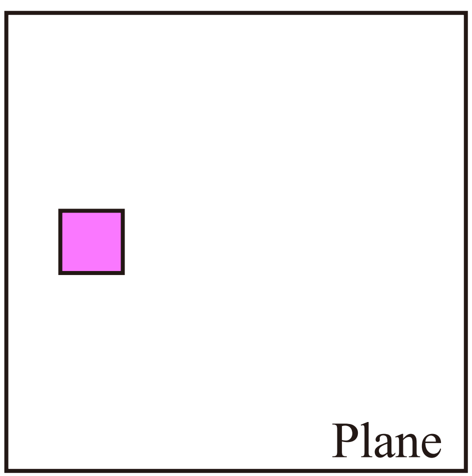
Technique to show th… e

Developed to check the overlaps efficiently.

Rotational Unfolding [T. Shio…

- Enumerating the path between any two faces by rolling a polyhedron.
- Checking the overlap of both end-faces of a path.





Technique to show the non-existence

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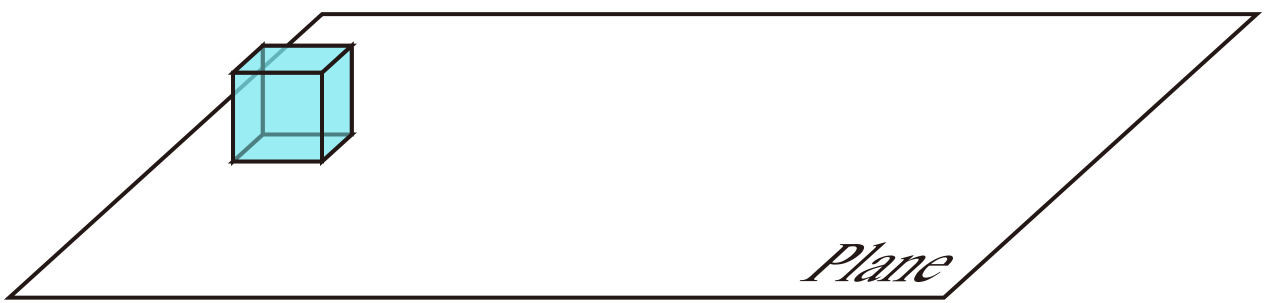

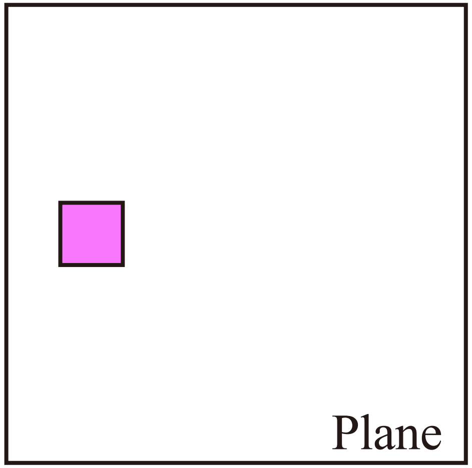

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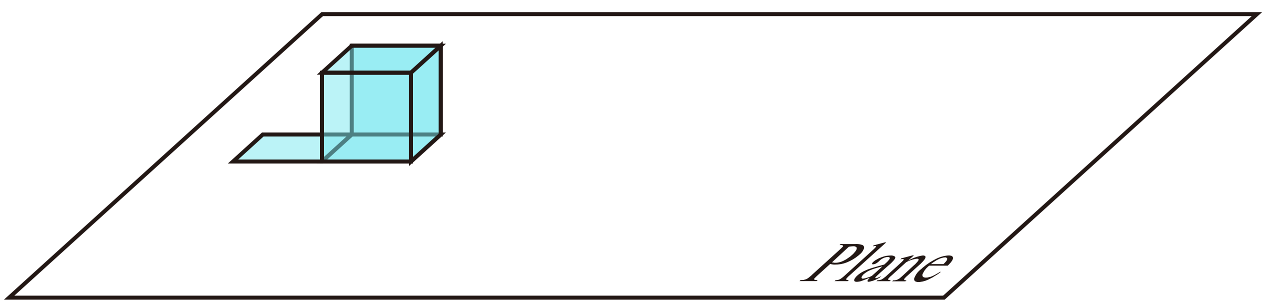

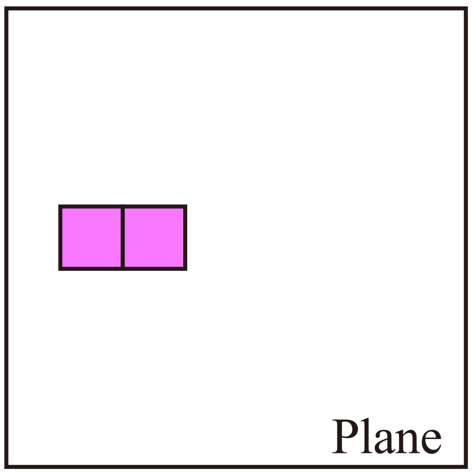

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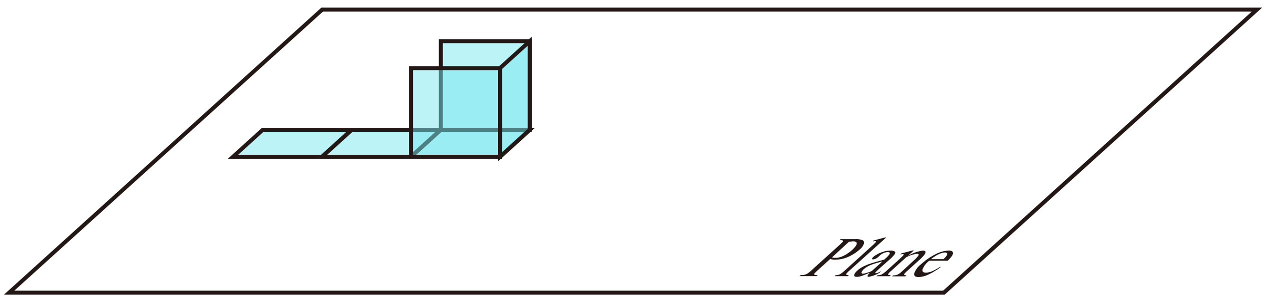

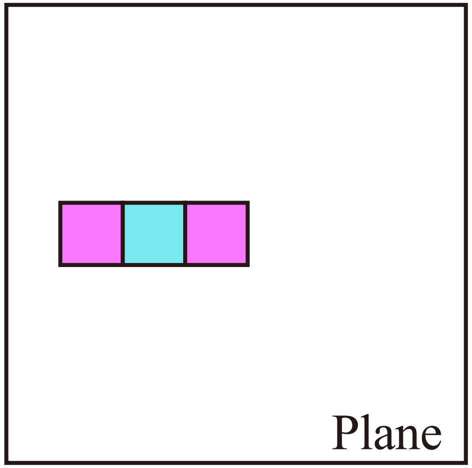

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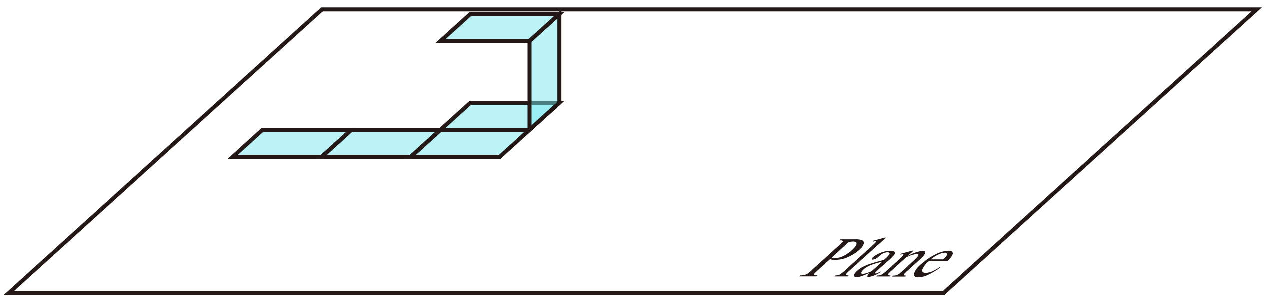

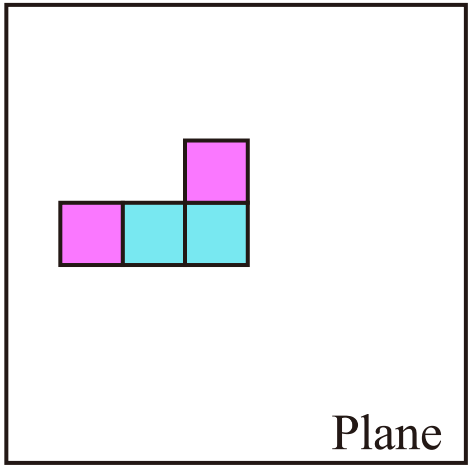

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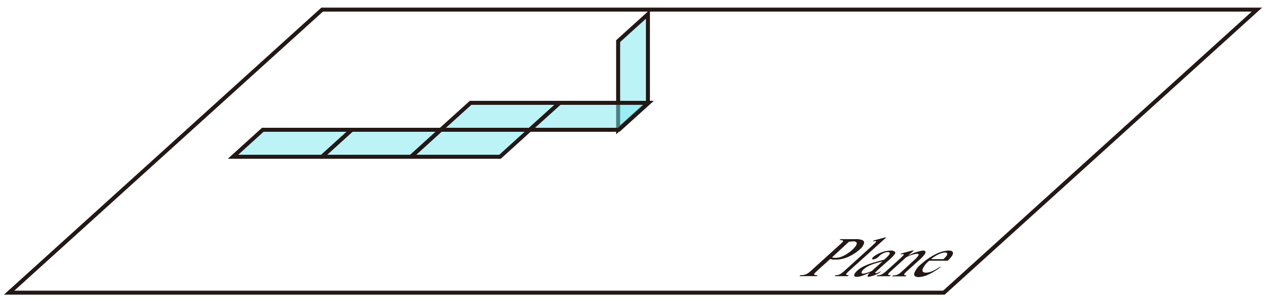

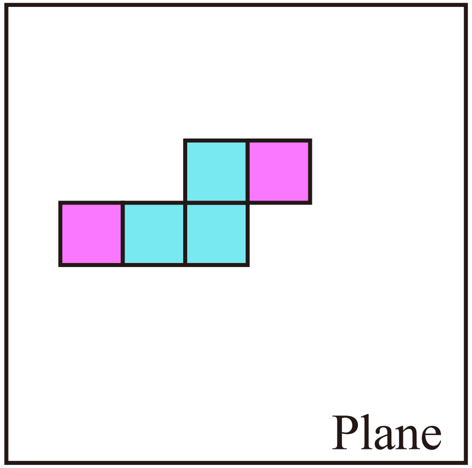

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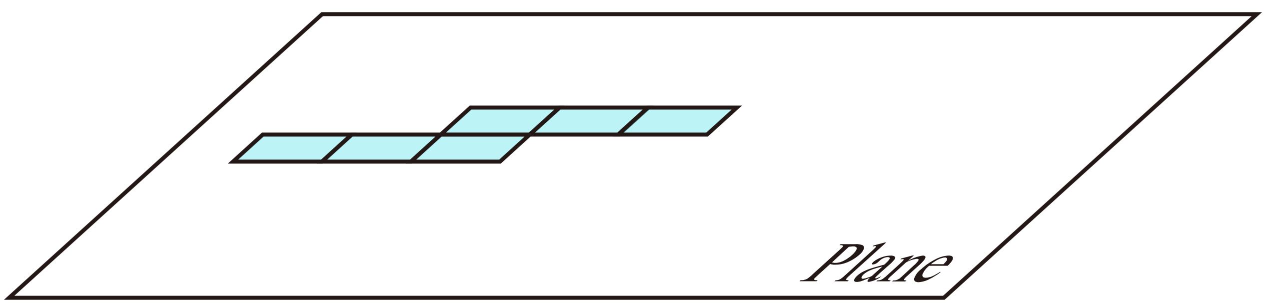

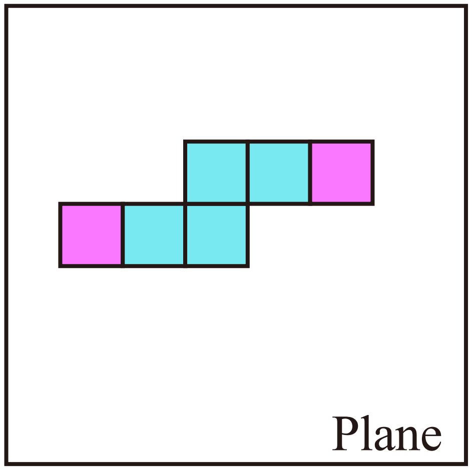

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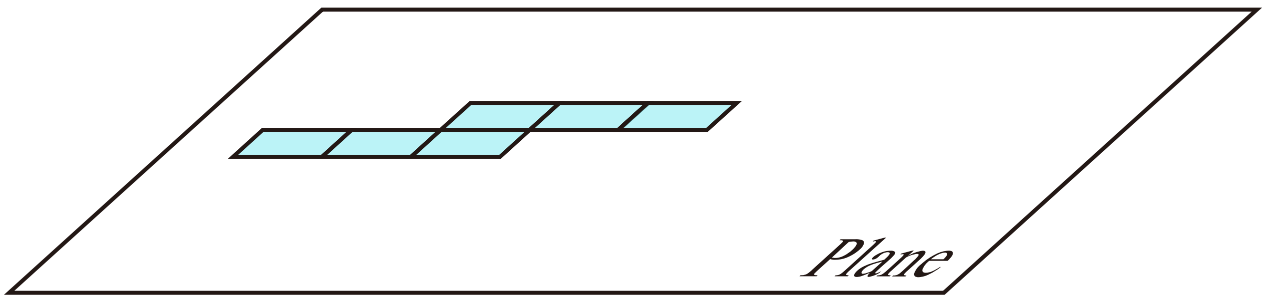
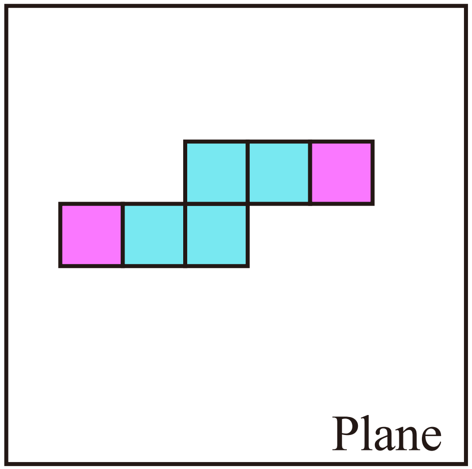


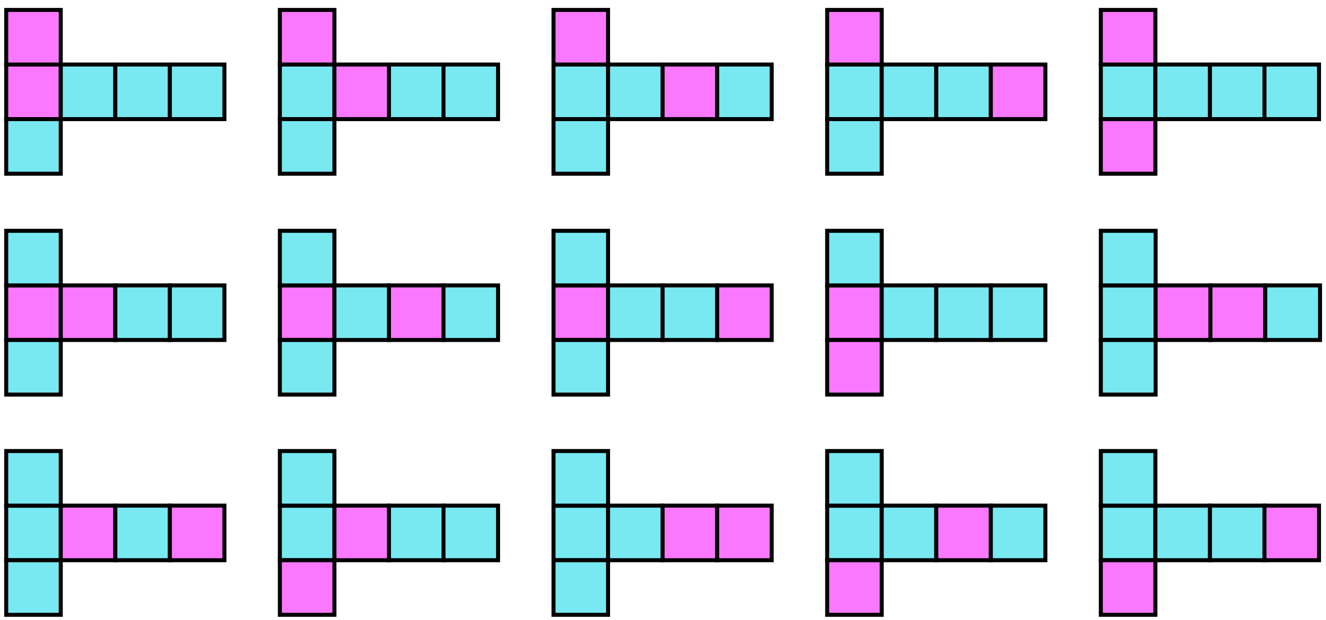


Q. Why only check the overlap of both end-faces in the path?

Technique to show the non-existence

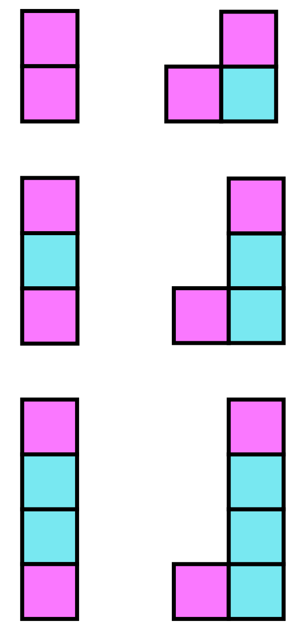
Lemma 1 [T. Shiota et al., 2023]

The path in the edge unfolding that connects two faces is one of the paths enumerated by rotational unfolding.



$_6C_2 = 15$ ways

Check all combinations of faces [T. Horiyama and W. Shoji, 2011]



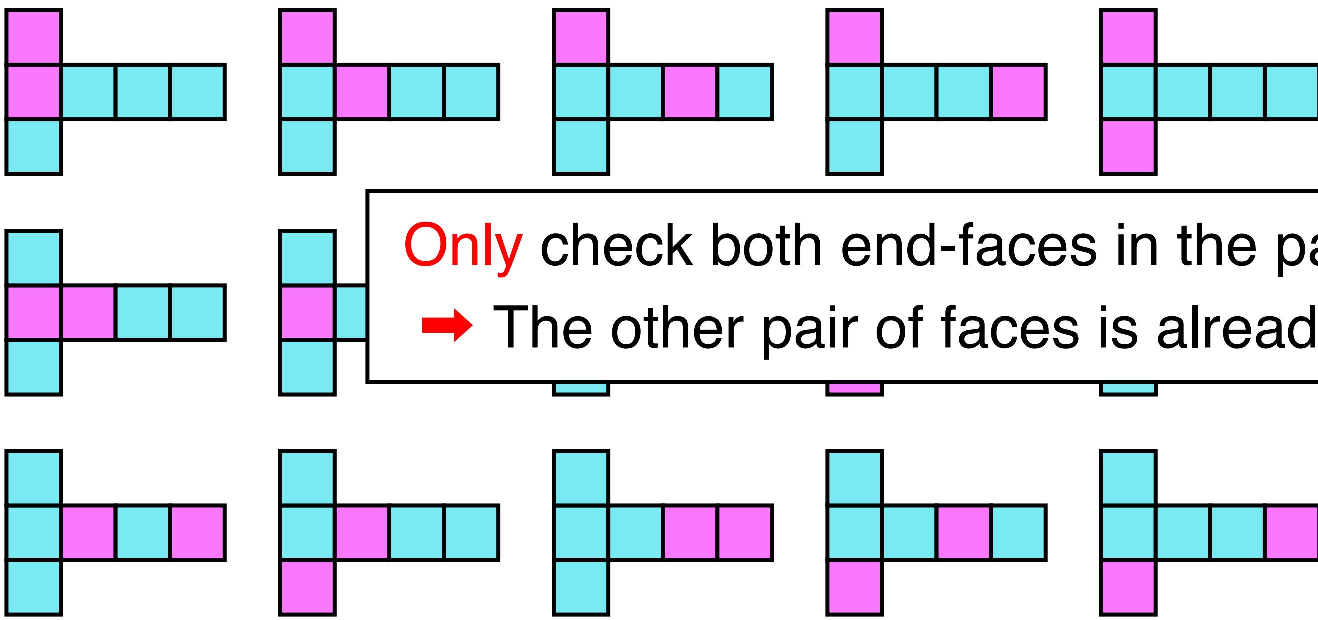
6 ways

Rotational unfolding

Technique to show the non-existence

Lemma 1 [T. Shiota et al., 2023]

The path in the edge unfolding that connects two faces is one of the paths enumerated by rotational unfolding.

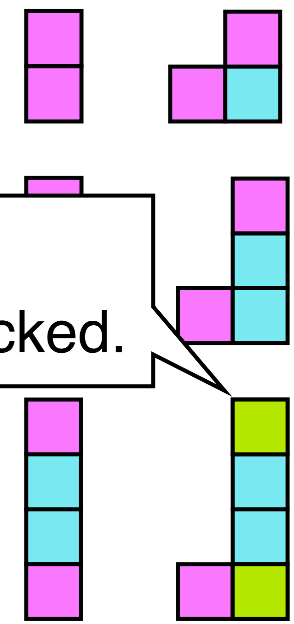


Only check both end-faces in the path.

➡ The other pair of faces is already checked.

$_6C_2 = 15$ ways

Check all combinations of faces [T. Horiyama and W. Shoji, 2011]



6 ways

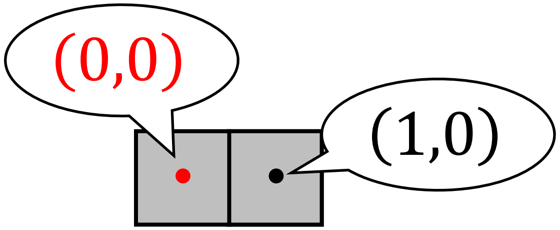
Rotational unfolding

Overlap check in lattice unfoldings

In rotational unfolding, we check for overlaps with each roll.

1. Set the center coordinates of one endpoint of the path to $(x, y) = (0,0)$.
2. While rolling the cuboid, sequentially compute the center coordinates of the other endpoint.

[Note] The length of one side of the cuboid is 1.



The computation process for the other endpoint's coordinates

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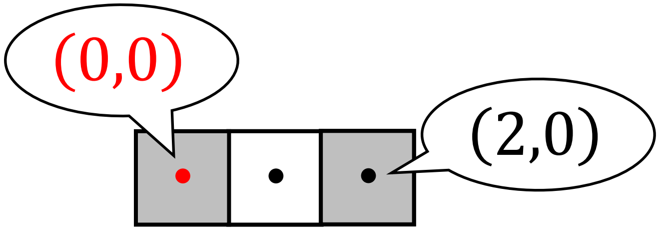

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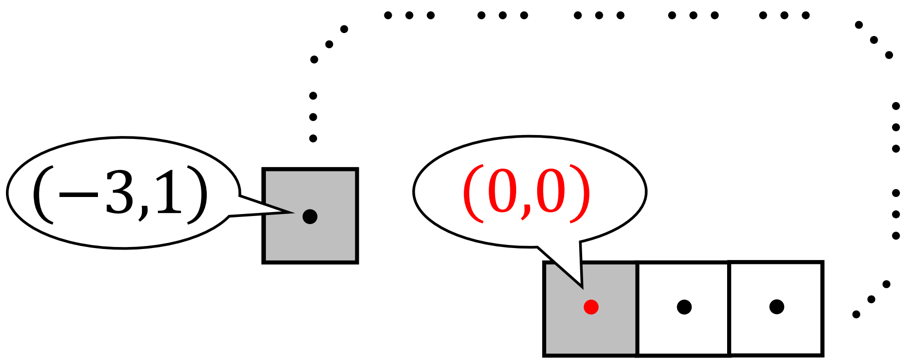

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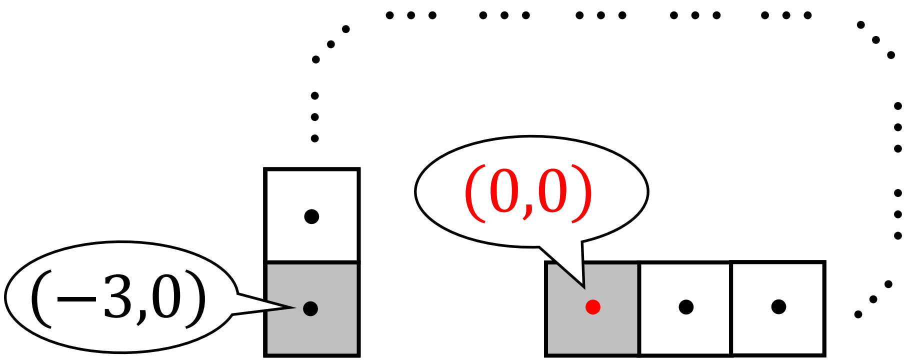

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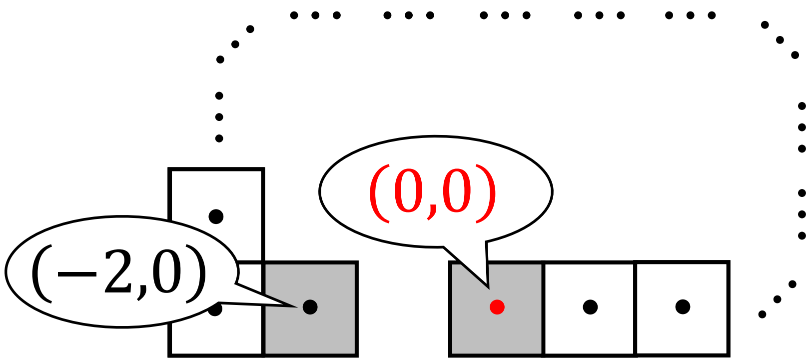


The computation process for the other endpoint's coordinates

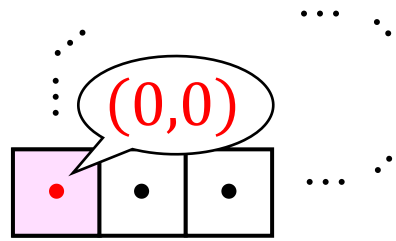
Overlap check in lattice unfoldings

The center coordinates of the other endpoint of the path are...

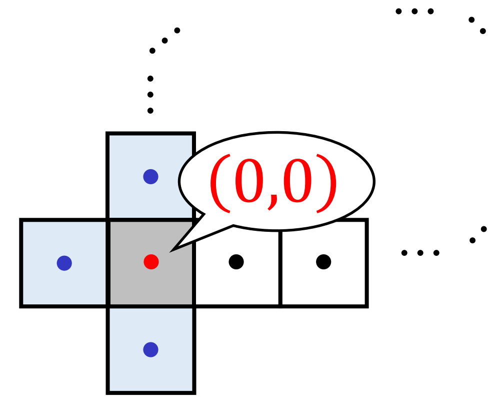
- $(0,0)$
  $\rightarrow$ Faces-in-touch

- $(0,1), (-1,0), (0,-1)$
  $\rightarrow$ Edges-in-touch

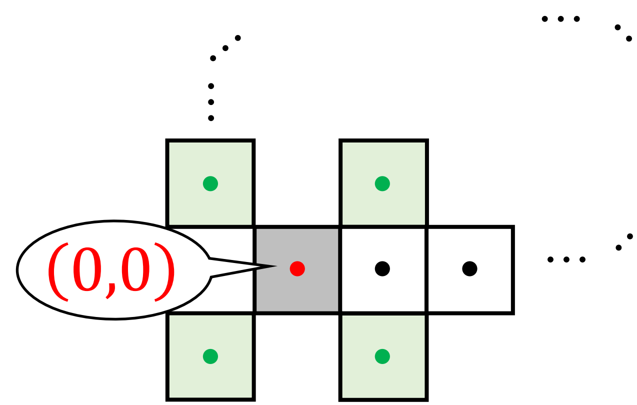
- $(1,1), (1,-1), (-1,-1), (-1,1)$
  $\rightarrow$ Vertices-in-touch



Faces-in-touch



Edges-in-touch



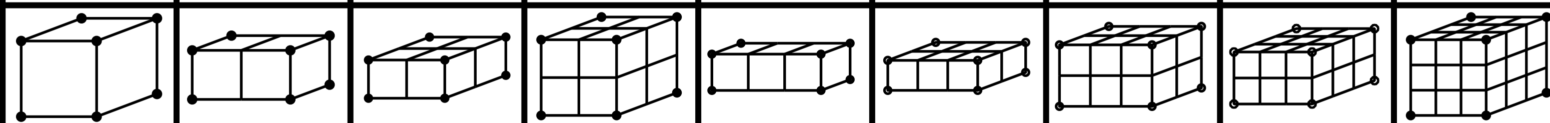
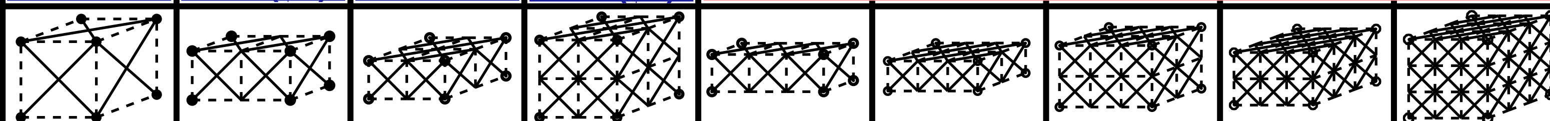
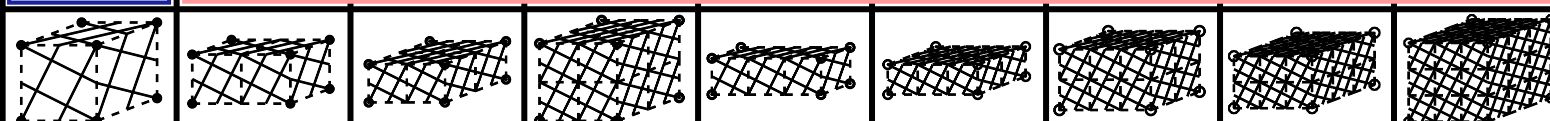
Vertices-in-touch

†1 : [R. Hearn, 2018] †2 : [H. Sugiura, 2018]

Background and our results

V	Vertices-in-touch
E	Edges-in-touch
F	Faces-in-touch

(a, b) ※ $\gcd(a, b) = 1$

(a,b)		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
$(1,0)$											$\cdots$
	V	No (Obvi.)	Yes	Yes		$(1\times1\times z)$-cuboid: found by [T. Uno, 2008] Otherwise: found by [J. Mitani et al., 2008]					
	E		No								
	F		No (†1)	No	No (†2)						
$(1,1)$											$\cdots$
	V	No	Yes								
	E										
	F										
$(2,1)$											$\cdots$
	V	Yes									
	E										
	F										
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
	V	Yes									
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	F										

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Background and our results

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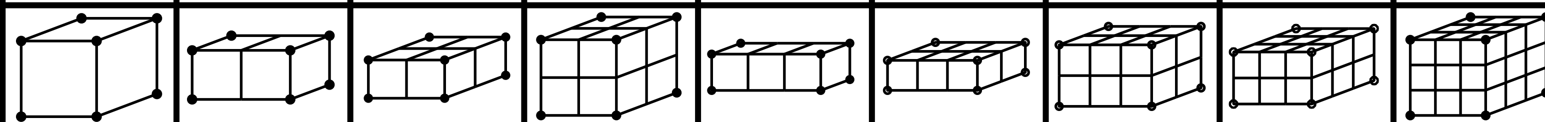
(a,b)		(x,y,z)									
		$(1,1,1)$	$(1,1,2)$	$(1,2,2)$	$(2,2,2)$	$(1,1,3)$	$(1,2,3)$	$(2,2,3)$	$(2,3,3)$	$(3,3,3)$	$\cdots$
$(1,0)$											$\cdots$
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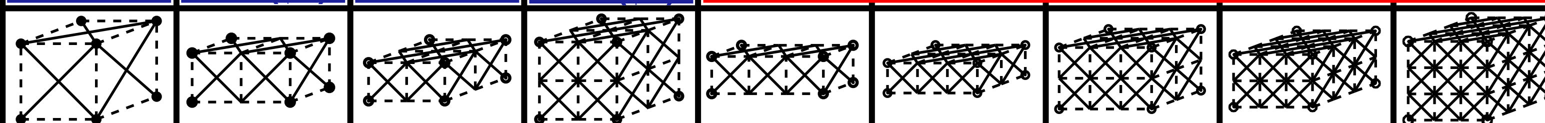
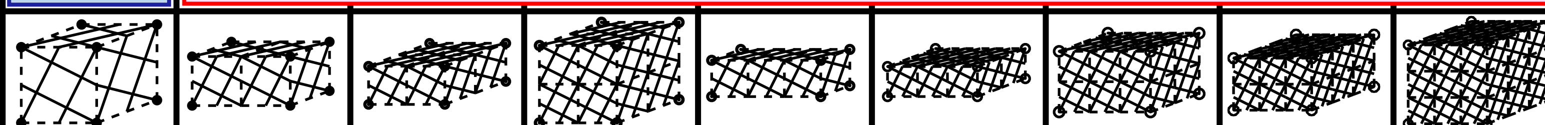
	F		**No (†1)**	**No**	**No (†2)**						
$(1,1)$											$\cdots$
	V	**No**	**Yes**								
	E										
	F										
$(2,1)$											$\cdots$
	V	**Yes**									
	E										
	F										
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
	V	**Yes**									
	E										
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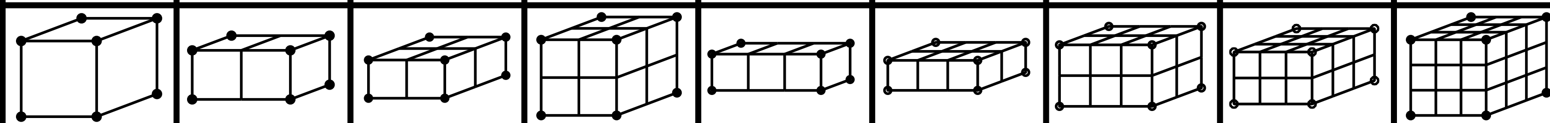
$(1,0)$											$\cdots$
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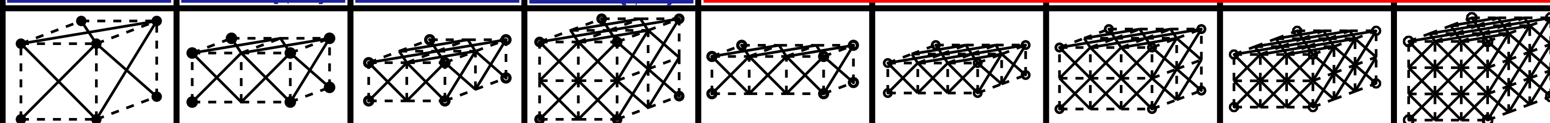
$(1,1)$											$\cdots$
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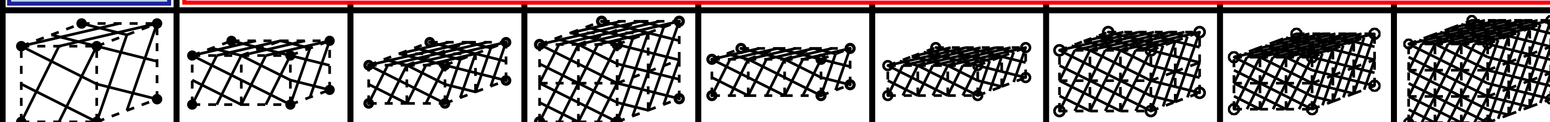
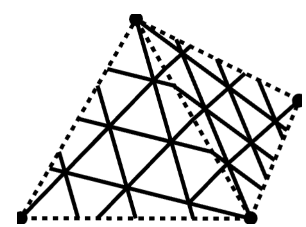
	E										
	F										
$(2,1)$											$\cdots$

Future work: Clarify the existence of overlapping unfolding for "tetrahedron" or "octahedron" that can be constructed from the triangular lattice.


Tetrahedron