Critical Sets of n-omino Sudoku

TAKASHI HORIYAMA, TONAN KAMATA, *HIRONORI KIYA, HIROTAKA ONO,TAKUMI SHIOTA, RYUHEI UEHARA, YUSHI UNO

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Sudoku

- A logic-based globally popular pencil puzzle.
- Also known as "Number Place."
- The objective: Fill a 9×9 grid with numbers so that each row, each column, and each of the nine 3×3 sub-grids contain all of the digits from 1 to 9.
- A solution **must be unique**.

	5	9				7	3	
			6			1		9
			4				5	
	3	6	1					5
	1	4	7		6	2	8	
2					4	6	7	
		4			3			8
5		3			7			
	9	2				3	6	

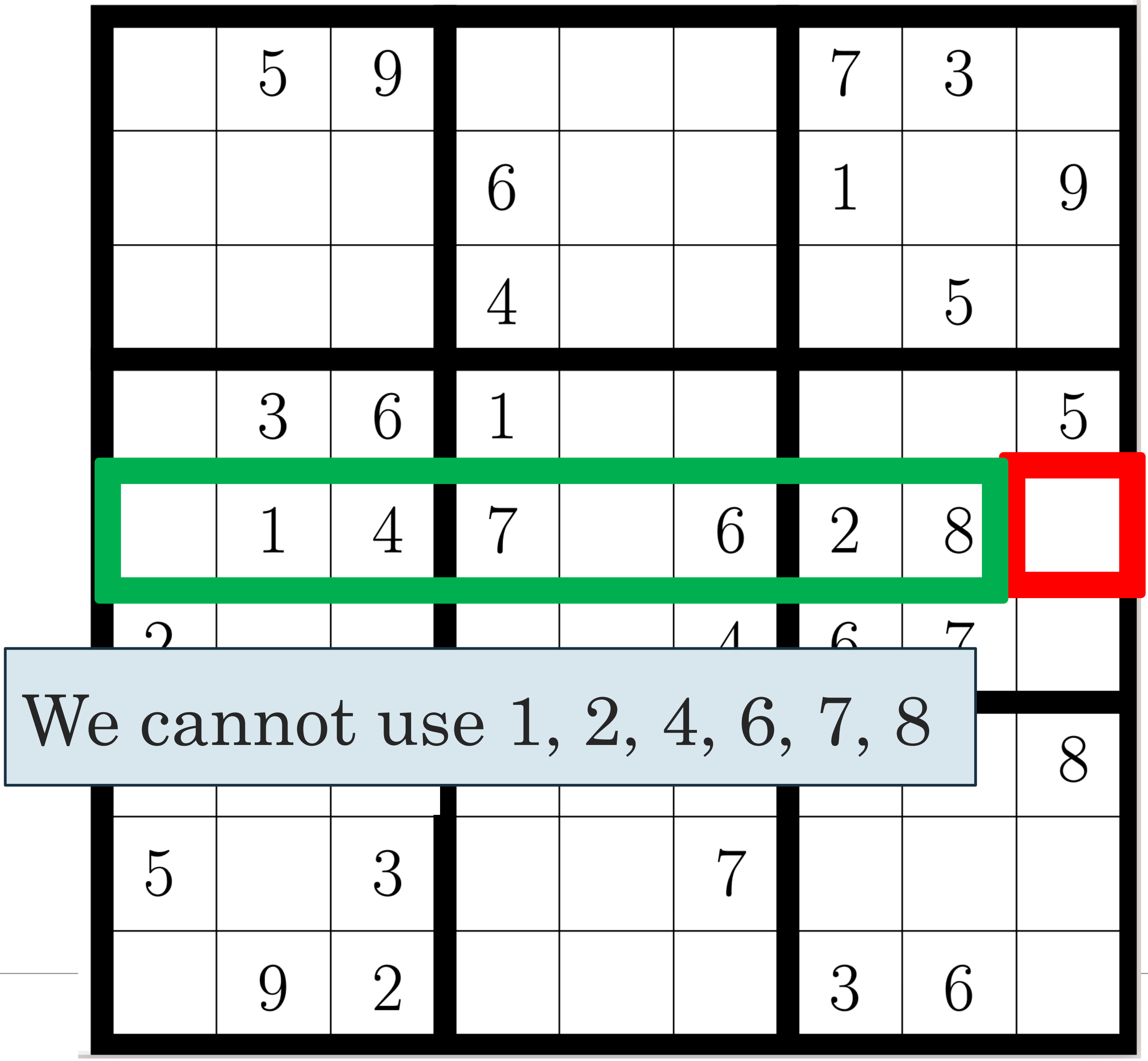
An example to solve Sudoku puzzle

	5	9				7	3	
			6			1		9
			4				5	
	3	6	1					5
	1	4	7		6	2	8	
2					4	6	7	
					3			8
5		3			7			
	9	2				3	6	

An example to solve Sudoku puzzle

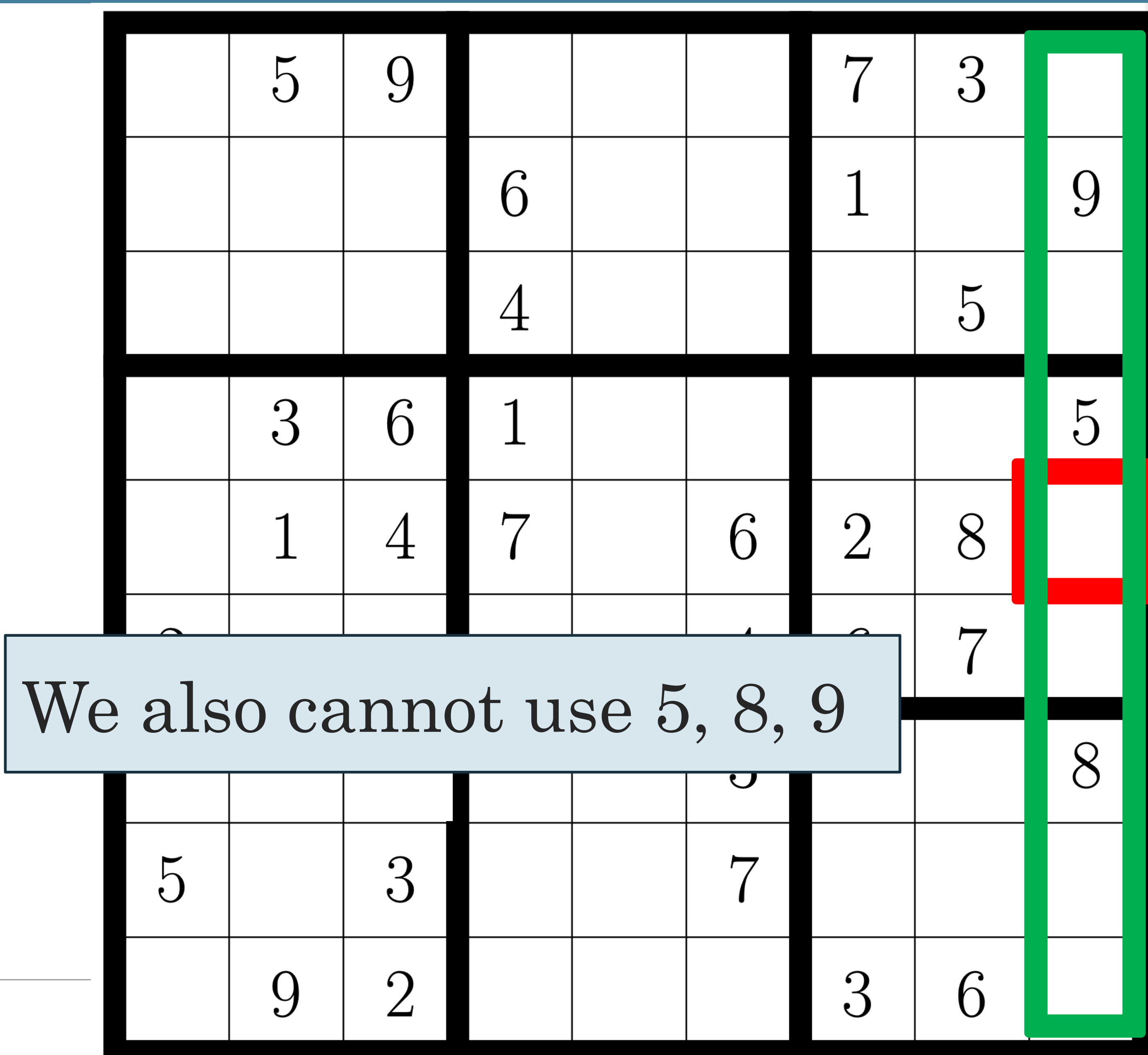
	5	9				7	3	
			6			1		9
			4				5	
	3	6	1					5
	1	4	7		6	2	8	
2					4	6	7	
					3			8
5		3			7			
	9	2				3	6	

An example to solve Sudoku puzzle

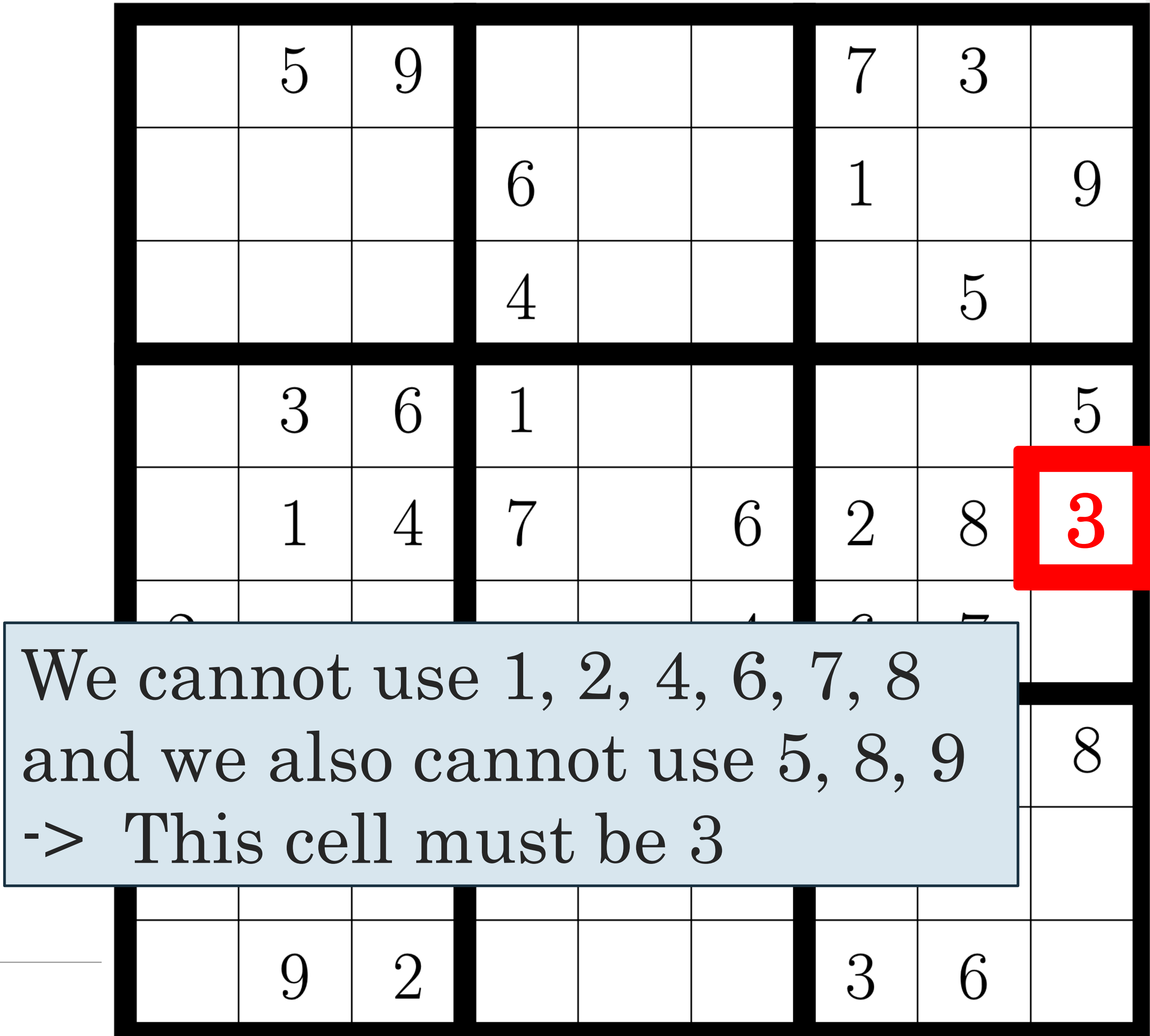


We cannot use 1, 2, 4, 6, 7, 8

An example to solve Sudoku puzzle



An example to solve Sudoku puzzle



An example to solve Sudoku puzzle

6	5	9				7	3	
4		8	6			1		9
3		1	4			8	5	
7	3	6	1					5
9	1	4	7	5	6	2	8	3
2	8	5			4	6	7	
1	4	7			3			8
5	6	3			7			
8	9	2				3	6	

An example to solve Sudoku puzzle

6	5	9				7	3	4
4	7	8	6			1	2	9
3	2	1	4			8	5	6
7	3	6	1			9	4	5
9	1	4	7	5	6	2	8	3
2	8	5			4	6	7	1
1	4	7			3	5	9	8
5	6	3			7	4	1	2
8	9	2				3	6	7

An example to solve Sudoku puzzle

6	5	9	8	1	2	7	3	4
4	7	8	6	3	5	1	2	9
3	2	1	4	7	9	8	5	6
7	3	6	1	2	8	9	4	5
9	1	4	7	5	6	2	8	3
2	8	5	3	9	4	6	7	1
1	4	7	2	6	3	5	9	8
5	6	3	9	8	7	4	1	2
8	9	2	5	4	1	3	6	7

Banned instance(multi solution)

6	5			1	2	7	3	4
4	7		6	3	5	1	2	
3	2	1	4	7			5	6
7	3	6	1	2			4	5
	1	4	7	5	6	2		3
2		5	3		4	6	7	1
1	4	7	2	6	3	5		
5	6	3			7	4	1	2
		2	5	4	1	3	6	7

One of solution

6	5	9	8	1	2	7	3	4
4	7	8	6	3	5	1	2	9
3	2	1	4	7	9	8	5	6
7	3	6	1	2	8	9	4	5
9	1	4	7	5	6	2	8	3
2	8	5	3	9	4	6	7	1
1	4	7	2	6	3	5	9	8
5	6	3	9	8	7	4	1	2
8	9	2	5	4	1	3	6	7

Another solution

6	5	8	9	1	2	7	3	4
4	7	9	6	3	5	1	2	8
3	2	1	4	7	8	9	5	6
7	3	6	1	2	9	8	4	5
8	1	4	7	5	6	2	9	3
2	9	5	3	8	4	6	7	1
1	4	7	2	6	3	5	8	9
5	6	3	8	9	7	4	1	2
9	8	2	5	4	1	3	6	7

One of solution

6	5	9	8	1	2	7	3	4
4	7	8	6	3	5	1	2	9
3	2	1	4	7	9	8	5	6
7	3	6	1	2	8	9	4	5
9	1	4	7	5	6	2	8	3
2	8	5	3	9	4	6	7	1
1	4	7	2	6	3	5	9	8
5	6	3	9	8	7	4	1	2
8	9	2	5	4	1	3	6	7

Another multi solution example

1	2	3	4	5	6	7	8	9
					2			
						3		

Critical sets

- Critical sets refer to the minimal subset of given clues that still ensure a **unique** solution.
- Their presence is pivotal in determining the solvability of a puzzle.
- In Sudoku, It is known the size of critical sets is 17 (It has been shown that with 16 or fewer clues, the solution is not unique).

				5	1			
	4						3	
	7		4	3		6		
5						1		2
2			8					
			3				7	
1		8						

The example of ensuring a unique solution (#clues = 17)

Latin square and its critical sets

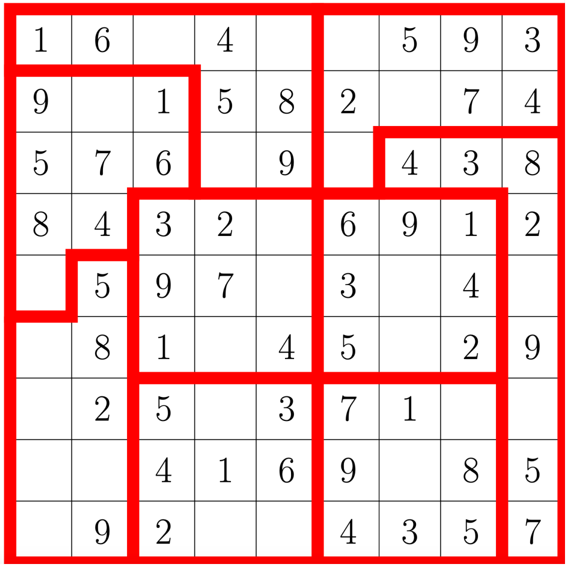
- When we remove the 3×3 sub-grids constraint from Sudoku, we call it (9×9)Latin square puzzle.
- With only row and column constraints, as in a Latin square, the known minimum number of clues required to ensure a unique solution is 20.
- It has not been proven whether or not a puzzle with 19 or fewer clues exists.

1	2	3	4					
2	3	4						
3	4							
4								
								5
							5	6
						5	6	7
					5	6	7	8

The example of ensuring a unique solution (#clues = 20)

Nonomino-Sudoku

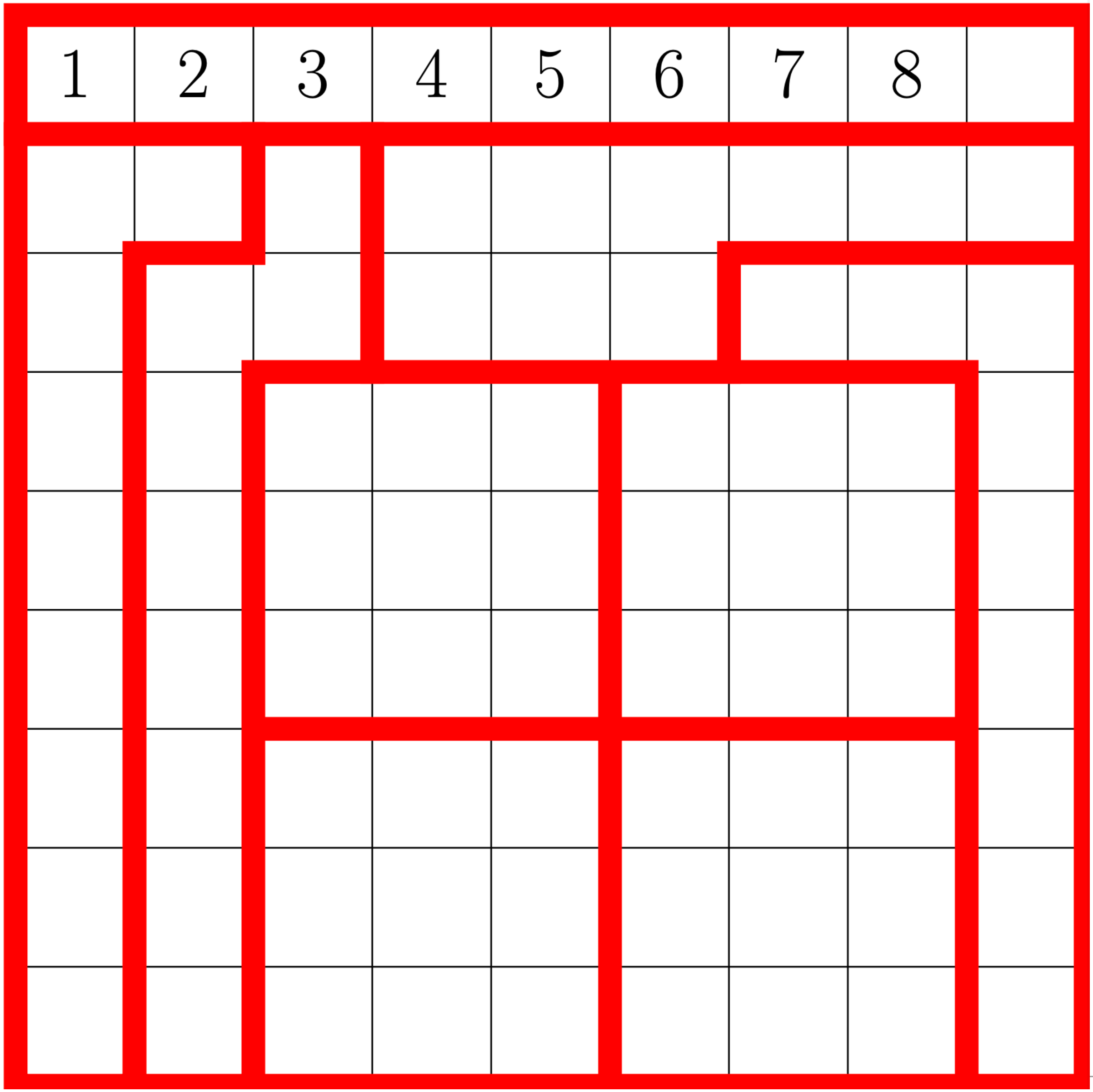
- We propose a puzzle where numbers 1-9 must be placed once in each row, column, and **each nonomino (a designated set of 9 adjacent cells).**
- We call it **Nonomino Sudoku.**
- What is the size of critical set?
- How should the segments be designated?



Theorem

Theorem 1

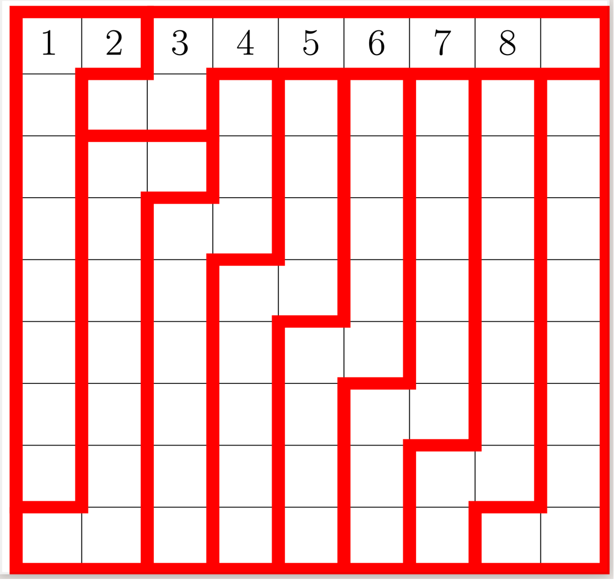
The size of a critical set of Nonomino Sudoku is 8



n-omino Sudoku

In this study, we further generalize Nonomino Sudoku to ``n-omino Sudoku'', which is defined on an $n \times n$ grid.

The objective: numbers 1 to n must be placed once in each row, column, and **each n-omino (a designated set of 9 adjacent cells).**

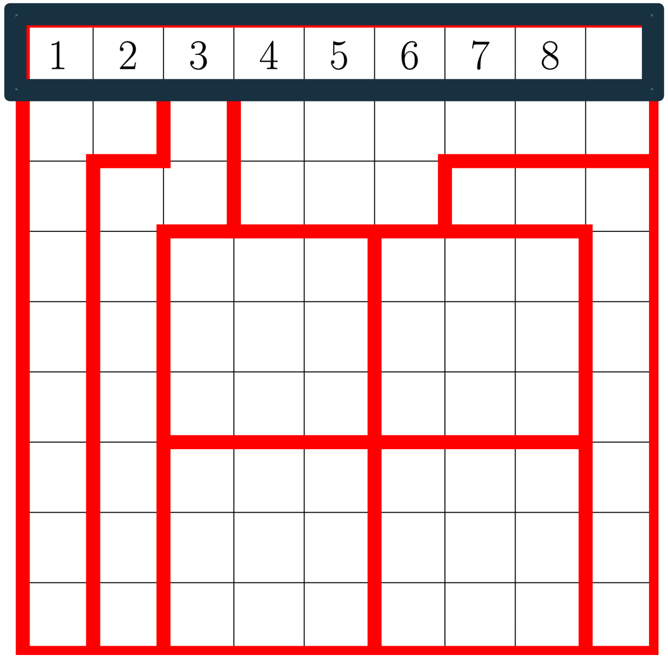


Theorem 2

The size of a critical set of nonomino Sudoku is $n - 1$.

Degeneracy

- Ordinary Sudoku has 27 constraints: 9 vertical, 9 horizontal, and 9 of the 3×3 sub-grids.
- The n–omino Sudoku is derived from the $n \times n$ Latin square's constraints by adding n constraints of n–omino. This results in a total of $3n$ constraints.
- We'll refer to the number of these added n–omino constraints that overlap with the Latin square constraints as the "degeneracy".



The example of Degeneracy = 1

Degeneracy and the size of clues

Our research focuses on the relationship between Degeneracy and the number of clues in Sudoku.

Corollary 1

There exists an n-omino Sudoku instance with degeneracy 0 or 1 such that its critical set has size $n - 1$.

Theorem 3

There are no n-omino Sudoku instance with degeneracy 3 or more such that its critical set has size $n - 1$.

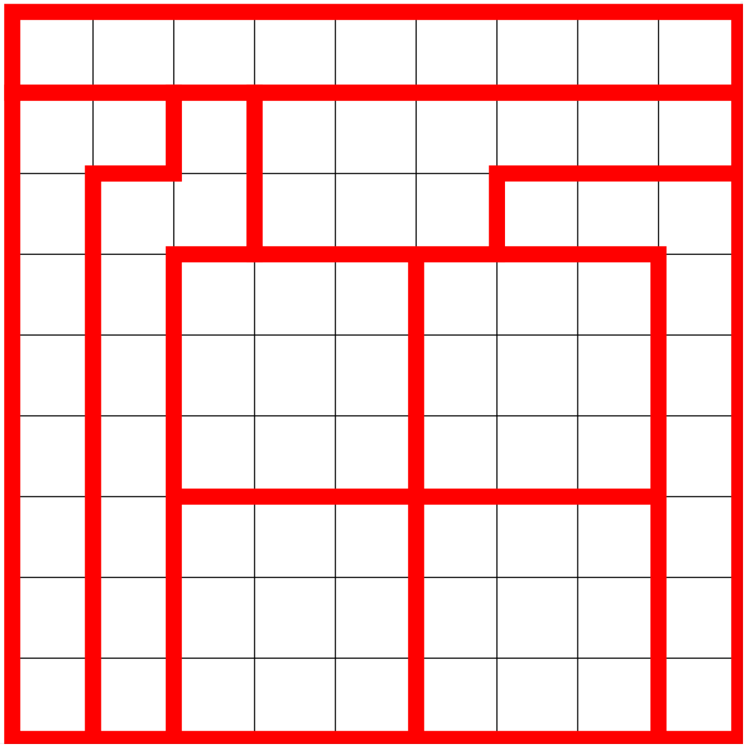
No solution without any clue

- We discussed the uniqueness of solutions.
- Intuitively, an instance with a small number of clues tends to have more solutions, or more clues tend to make a solution unique.
- Thus, we might expect that an instance with no clue has many solutions.
- However, In n -omino Sudoku, depending on the placement of the n-ominoes, there may be no solution.

No clue instance with no solution

Theorem 3

Let k and n be positive integers satisfying $n \leq 2k + 4$. Then, there is an n-omino Sudoku instance with degeneracy k and no clue such that it has no solution.



No solution without any clue.