A Characterization of the Overlap-free Polyhedra

Tonan Kamata, Takumi Shiota, Ryuhei Uehara

Abstract: *For a polyhedron Q, a polygon obtained by cutting the edges or faces of Q is called a general unfolding. A general unfolding may have a self-overlap or self-intersection on the boundary which depends on the way of unfolding. It is established by [Aronov and O'Rourke] and [Sharir and Schorr] that any convex polyhedron satisfies the property that at least one general unfolding has no overlap. This research focuses on a dual property in which any general unfolding has no overlap, called overlap-free. We show that a polyhedron is overlap-free if and only if it is a stamper, which is a notion introduced by Akiyama. This means that if a polyhedron is not a stamper, at least one general unfolding has an overlap. We prove it in a constructive way.*

1 Introduction

The unfolding of polyhedra has been used for a long time as a method to represent the surface structure of polyhedra. Its origin is said to be in the work "Underweysung der messung mit dem zirckel un richt scheyt" by the 15th-century painter A. Dürer. In this work, Dürer represented polyhedra using diagrams that arranged each face according to their adjacency relationships. The diagrams used by Dürer can be considered as polygons obtained by cutting and unfolding the surface of the polyhedra along the edges, which are now referred to as *edge unfoldings*. The edge unfoldings of polyhedra can, depending on how they are cut, result in faces touching or overlapping each other (Figure 1).

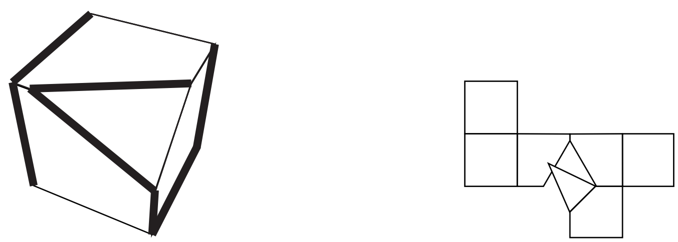

Figure 1: *According to [Namiki and Fukuda 93], polyhedra whose edge unfoldings have overlaps*

However, in Dürer's work, all polyhedra are represented with edge unfoldings that do not overlap. Focusing on this point, the 20th-century mathematician Shephard proposed the following conjecture [Shephard 75]:

Conjecture 0.1. *Every convex polyhedron has an edge unfolding that does not overlap.*

This conjecture remains unsolved to this day and is being studied from various perspectives. One approach involves considering general unfoldings that allow cuts in places other than the edges during the unfolding process. With this extended method of unfolding, the following theorem holds:

Theorem 1. *Every convex polyhedron has a non-overlapping general unfolding.*

This theorem is proven by constructing a method of general unfolding that guarantees no overlaps. Specific methods of unfolding include the source unfolding by Sharir and Schorr [Sharir and Schorr 86] and the star unfolding by Aronov and O'Rourke [Aronov and O'Rourke 92].

While the above result concerns the existence of non-overlapping unfoldings, it is also possible to consider propositions about universality, namely, the property that "all general unfoldings do not overlap." When a polyhedron satisfies this property, it is defined as *Overlap-free*. This study provides a complete characterization for the class of polyhedra that are Overlap-free. Specifically, it uses a property called *Stamper* to show that this property is a necessary and sufficient condition for being Overlap-free. Stamper, as will be discussed later, is a concept proposed by Akiyama et al. to capture the tiling possibility of an unfolding diagram. Akiyama et al. provide a specific classification of polyhedra that are Stampers. Using this, it was possible to obtain a complete classification of Overlap-free polyhedra.

2 Preliminaries

2.1 Basic Definitions

When the boundary of a bounded convex subset in $\mathbb{R}^3$ is composed of a finite number of polygons, this boundary is called a *convex polyhedron*, and the polygons composing the boundary are referred to as *faces*. In this study, convex polyhedra with a volume of 0, that is, shapes that only have two congruent polygons as their faces, are also included in the definition of polyhedra. Such polyhedra are called *doubly covered polygons*.

An *unfolding* of a polyhedron Q is a connected and flat polygon obtained by cutting open the surface of Q. The set of vertices of a polyhedron is denoted by $V(Q)$. For a vertex $v \in V(Q)$, let $\sigma(v)$ be the total sum of the internal angles at each face meeting at vertex v, which is referred to as the *co-curvature*. Regarding the value of $\sigma(v)$, the following theorem, known as *Descartes' theorem*, holds:

Lemma 1.1. *For any convex polyhedron Q,*

$$\sum_{v \in V(Q)} (2\pi - \sigma(v)) = 4\pi.$$

2.2 The Stamper Property of Polyhedra

This section discusses the property known as *Stamper* on polyhedra, including its definition and characterization. The Stamper property was defined by Akiyama et al. in [Akiyama and Matsunaga 18, Akiyama and Matsunaga 20, Akiyama and Nakamura 00]. For proofs of lemmas and specific examples, refer to the respective publications. In this paper, we use the following definition based on [Kamata et al. 22]. For a polyhedron Q, we consider a sequence $s = (f_1, e_1, f_2, e_2, \ldots, f_k)$ which line up faces f_i and edges e_i alternately. Here, we assume that neighboring two faces f_i and f_{i+1} are distinct faces sharing an edge e_i. We use the terminology "stamp Q along s" for the following operation in $\mathbb{R}^3$:

- As the initial position, place Q on the xy plane such that f_1 becomes the bottom face.
- Rotate Q around the edge shared by f_i and f_{i+1}, from a state where the bottom face is f_i, so that f_{i+1} becomes the new bottom face. Repeat this operation for all elements of s.

Here, let $\partial(Q)$ be the set of points on the surface of Q. When polyhedron Q is stamped along s, let $A_s \subset \mathbb{R}^2$ be the set of points that are included in the bottom face of Q at least once. Stamping generally derives a many-to-many correspondence between A_s and $\partial(Q)$.

Here, we introduce the following notation:

Definition 1.1. *A polyhedron Q is said to be a* stamper *if, for any sequence s, the correspondence derived from stamping become a mapping from A_s to $\partial(Q)$. In other words, each point on A_s uniquely corresponds to a single point in $\partial(Q)$.*

For example, we consider stamping a cube in the order of A, C, D, E, A, C, D after labeling each face with A to F as shown in Figure 2. In this case, a single point on A_s corresponds to different vertices of Q between the first bottom face A and the last bottom face D. This indicates that the cube is not a stamper. On the other hand, for a regular tetrahedron, the point corresponding to a single point on the plane is unique in any stamping path (Figure 3). Therefore, the regular tetrahedron is a stamper.

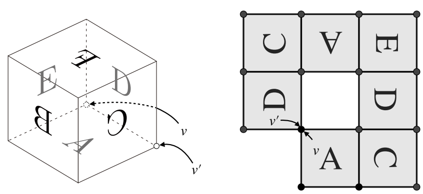

Figure 2: *Stamping for a cube*

For the stamper property, the following characteristics are known:

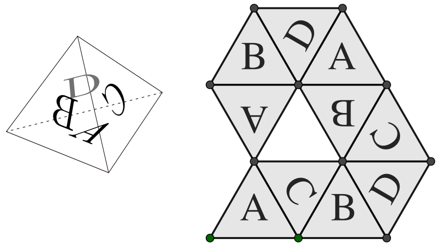

Figure 3: *Stamping for a regular tetrahedron*

Lemma 1.2. *For a convex polyhedron Q to be a Stamper, it is equivalent to being included in one of the following classes:*

- *Tetramonohedra; tetrahedra whose all faces are congruent*
- *Doubly covered right isosceles triangles; doubly covered polygons with two congruent right isosceles triangles as faces*
- *Doubly covered regular triangles; doubly covered polygons with two congruent regular triangles as faces*
- *Doubly covered half-regular triangles; doubly covered polygons with two congruent half-regular triangles as faces*

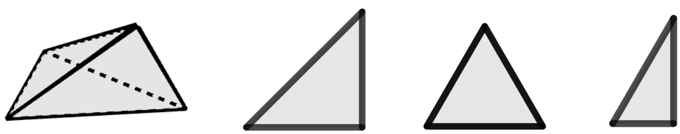

Figure 4: *Classes of polyhedra that satisfy the stamper property, from left to right: tetramonohedra, doubly covered right isosceles triangles, doubly covered regular triangles, doubly covered half regular triangles*

3 Results

This paper demonstrates the following theorem:

Theorem 2. *For any convex polyhedron Q, Q being a Stamper is a necessary and sufficient condition for Q to be Overlap-free.*

3.1 Proof of Necessity

Lemma 2.1. *If Q is a Stamper, then any unfolding of Q does not have overlaps.*

This lemma can be derived from the results of [Akiyama and Matsunaga 18, Akiyama and Matsunaga 20, Akiyama and Nakamura 00], but this paper provides a separate proof to clarify its relationship with overlap-free properties.

Proof. Assume that a certain unfolding P of Q has an overlap. Place P on $\mathbb{R}^2$ and take a point $p \in \mathbb{R}^2$ within the region where P overlaps. At this time, more than two points on the surface of Q are placed at the location of p. Denote these as q, q'. Now, consider stamping from the state where the face containing q becomes the bottom face to the state where the face containing q becomes the bottom face again. This stamping is uniquely determined [Kamata et al. 22]. In this Stamping, since p corresponds to two different points q, q', the derived correspondence is not a mapping. Therefore, Q is not a Stamper. □

3.2 Proof of Sufficiency

Before proving sufficiency, we present the following lemma:

Lemma 2.2. *When a polyhedron Q is not a stamper, there exists at least one vertex v that satisfies one of the following conditions:* $(A): \pi < \sigma(v)$, $(B): \frac{2\pi}{3} < \sigma(v) < \pi$, $(C): \frac{\pi}{2} < \sigma(v) < \frac{2\pi}{3}$.

Proof. Let n be the number of vertices of the polyhedron Q, and assume that Q is not a stamper. Note that if the co-curvature of all vertices of Q are composed only of $\pi, \frac{2\pi}{3}, \frac{\pi}{2}$, then Q is a stamper.

(CASE 1, when $n \geq 4$): From Lemma 1.1, the total sum of co-curvatures is at least 4π. If all vertices' co-curvatures were equal to π, Q would be a tetramonohedron, so there must be at least one vertex whose co-curvature is greater than π. Therefore, there must exist at least one vertex that satisfies condition (A).

(CASE 2, when $n = 3$): Let the three vertices of Q be v, v', v''. From Lemma 1.1, the total sum of co-curvatures is $\sigma(v) + \sigma(v') + \sigma(v'') = 2\pi$. Assume that none of the vertices satisfy condition (A). Further, divide the cases based on whether there exists a vertex with a co-curvature of π.

(CASE 2-1, when one vertex's co-curvature is π): Without loss of generality, assume $\sigma(v) = \pi, \sigma(v') < \sigma(v'')$. From Lemma 1.1, $\sigma(v') + \sigma(v'') = \pi$. By the assumption that Q is not a Stamper, it is not possible for both $\sigma(v')$ and $\sigma(v'')$ to be $\frac{\pi}{2}$. Therefore, v'' satisfies condition (B).

(CASE 2-2, when all co-curvatures are less than or equal to π): Without loss of generality, assume $\sigma(v) < \sigma(v') < \sigma(v'')$. From Lemma 1.1, $\sigma(v) + \sigma(v') + \sigma(v'') = 2\pi$. Given the assumption that Q is not a Stamper, it is not possible for $\sigma(v), \sigma(v'), \sigma(v'')$ to all be $\frac{2\pi}{3}$. Since $\sigma(v) + \sigma(v') + \sigma(v'') = 2\pi$, $\sigma(v'')$ must be truly greater than $\frac{2\pi}{3}$. Therefore, $\sigma(v'')$ satisfies condition (C). □

Lemma 2.3. *If Q is not a Stamper, then there exists an unfolding of Q that has overlaps.*

Proof. According to Lemma 2.2, a polyhedron must have at least one vertex that satisfies either condition (A), condition (B), or condition (C), implying that it has a vertex with co-curvature of at least $\pi/2$. Therefore, it suffices to show that for each condition (A) through (C), it is possible to unfold the neighborhood of a vertex satisfying any of these conditions in such a way that it results in overlaps. Consider a vertex v of the polyhedron Q that meets one of these conditions. Take a sufficiently small circle centered at v on the surface of Q, and cut v out of Q along the edge of this circle. This produces a sector F obtained by cutting open the cone, which has no bottom face, along a line from v towards the circumference (Figure 5). By definition, the central angle of F matches the co-curvature $\sigma(v)$ at v. Place F on the xy plane with its center point at the origin and the cut edge of the cone along the x-axis.

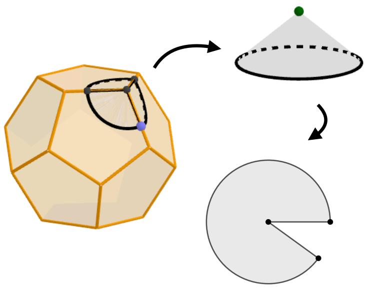

Figure 5: *Cutting out of the cone*

Case where v satisfies condition A: In this case, a sector F with a central angle greater than or equal to π and less than 2π is obtained (refer to Figure 6. For the sake of illustration, a regular decagon is used instead of a cone in the figure).

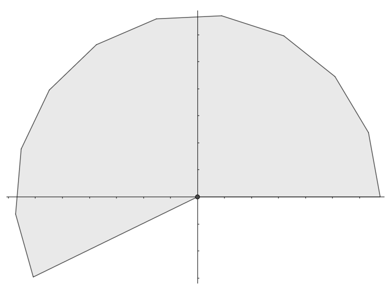

Figure 6: *Sector F obtained from a vertex satisfying condition (A)*

Take a circle C with center at the origin O and radius ε, and let the intersection points with F's radius be A, A'. Also, place a point B on the y-axis at a distance s from the origin (Figure 7 left). Cut out triangle OBA from F and glue it such that OA aligns with OA' to form shape F'. The point corresponding to B in the glued triangle is denoted as X. The coordinates of X can be expressed as $(s\cos(\sigma(v)+\frac{\pi}{2}), s\sin(\sigma(v)+\frac{\pi}{2}))$. Since OA and OA' correspond to the same point on the cone, F' also represents a partial unfolding of Q. Next, divide F with a line passing through the hypotenuse BA' and glue it similarly on the cone (Figure 7 right). This process results in a new unfolding F'' (Figure 8).

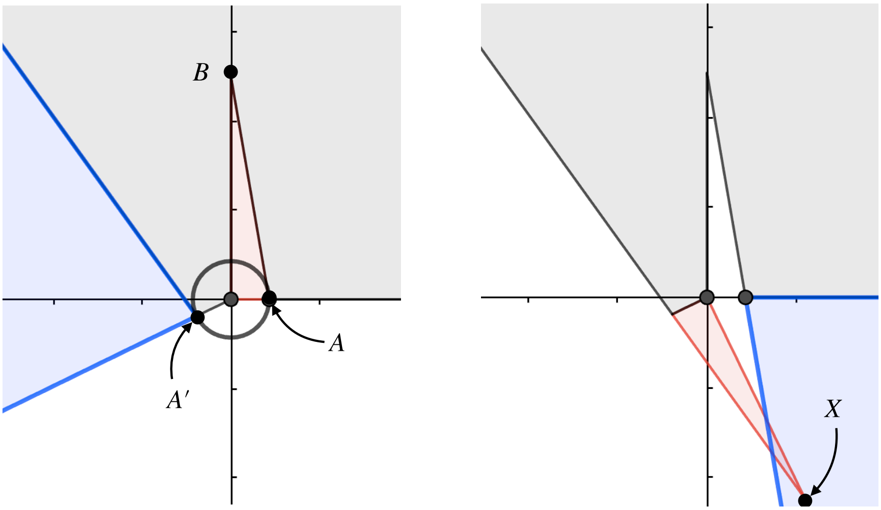

Figure 7: *Design of the cut line*

To demonstrate that the unfolding has overlaps, it suffices to show that there exist values of s, ε such that X lies above the line AB. The equation of line AB can be expressed as:

$$y = s\left(\frac{-1}{\varepsilon}x + 1\right)$$

The condition for point X to be above this line is:

$$s \cdot \sin\left(\sigma(v) + \frac{\pi}{2}\right) > s\left(\frac{-1}{\varepsilon}\left(s \cdot \cos\left(\sigma(v) + \frac{\pi}{2}\right) + 1\right)\right)$$

This inequality can be transformed, given that both s and ε are positive, to:

$$\varepsilon \cdot \sin\left(\sigma(v) + \frac{\pi}{2}\right) > -s \cdot \cos\left(\sigma(v) + \frac{\pi}{2}\right) + \varepsilon$$

$$s \cdot \cos\left(\sigma(v) + \frac{\pi}{2}\right) > \varepsilon \cdot \sin\left(\sigma(v) + \frac{\pi}{2}\right) + \varepsilon$$

Further, since $\pi < \sigma(v) < 2\pi$, it can be transformed to:

$$s > \varepsilon \cdot \frac{\sin\left(\sigma(v) + \frac{\pi}{2}\right) + 1}{\cos\left(\sigma(v) + \frac{\pi}{2}\right)} \tag{1}$$

Therefore, for any $\sigma(v)$ satisfying $\pi < \sigma(v) < 2\pi$, by choosing s such that point B fits within F and taking ε sufficiently small, an unfolding with overlaps can be obtained.

Case where v satisfies condition B: In this case, a sector F with a central angle greater than or equal to $\frac{2\pi}{3}$ and less than π is obtained (Figure 9).

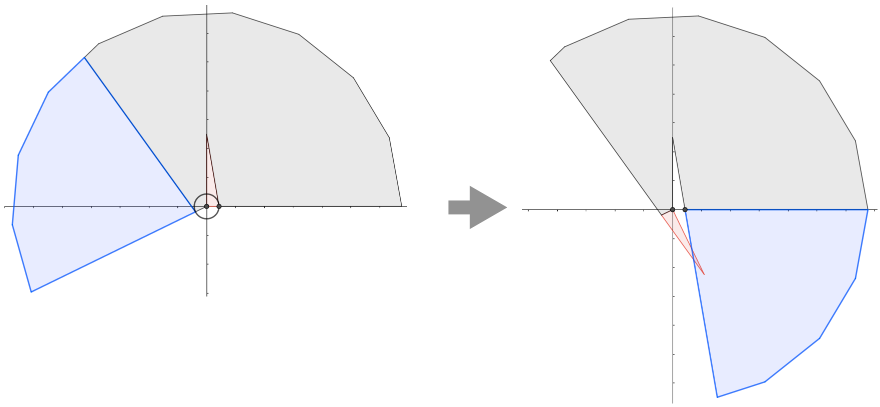

Figure 8: *Transformed unfolding F''*

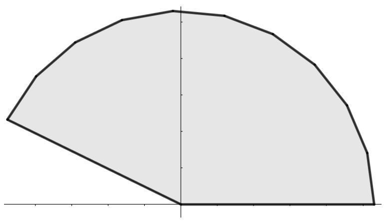

Figure 9: *Sector F obtained from a vertex satisfying condition (B)*

Here, take a circle C_1 with radius ε and a circle C_2 with radius 2ε, both centered at the origin O. Let the intersection points of each C_i with the radius of F be A_i, A'_i. Additionally, draw a line l_1 that passes through the origin and makes an angle of $\frac{2\pi}{3}$ with the x-axis, and place a point B_1 on l_1 at a distance s from O. Then, draw a line parallel to l_1 passing through A'_1 as l_2, and a line parallel to line A_1B_1 passing through A_2 as l_3, with their intersection point denoted as D (Figure 10 left).

Next, cut out triangle OB_1A_1 from F and glue it such that OA_1 aligns with OA'_1, and then cut F along two line segments A'_1D and DA_2, gluing them so that A_1A_2 aligns with $A'_1A'_2$ (Figure 10 right). At this point, denote the point corresponding to B_1 in the glued triangle OB_1A_1 as X. The coordinates of X become $\left(s\cos\left(2\sigma(v)+\frac{\pi}{3}\right), s\sin\left(2\sigma(v)+\frac{\pi}{3}\right)\right)$.

This results in a partial unfolding of Q, F', similar to the case with condition A (Figure 11).

To demonstrate that the unfolding has overlaps, it suffices to show that there exist values of s, ε such that X lies above the line A_2D. At this point, the equation of line A_2D can be expressed as:

$$y = \frac{s\sqrt{3}}{s+2\varepsilon}(-x+2\varepsilon)$$

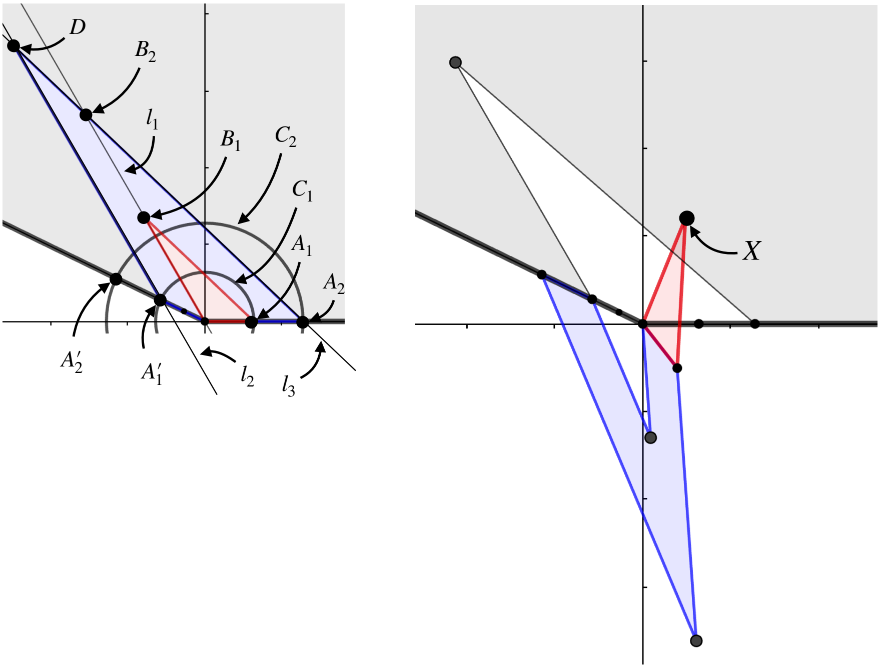

Figure 10: *Design of the cut line*

The condition for point X to be above this line is expressed as:

$$s \cdot \sin\left(2\sigma(v) + \frac{\pi}{3}\right) > \frac{s\sqrt{3}}{s+2\varepsilon}\left(-s \cdot \cos\left(2\sigma(v) + \frac{\pi}{3}\right) + 2\varepsilon\right)$$

Given that both s and ε are positive, this inequality can be transformed as follows:

$$\begin{aligned}&(s+2\varepsilon)\sin\left(2\sigma(v) + \frac{\pi}{3}\right) \\ &> \sqrt{3}\left(-s \cdot \cos\left(2\sigma(v) + \frac{\pi}{3}\right) + 2\varepsilon\right)\end{aligned}$$

Further, by rearranging in terms of s, the equation can be transformed as follows.

$$\begin{aligned}&s \cdot \left(\sin\left(2\sigma(v) + \frac{\pi}{3}\right) + \sqrt{3}\cos\left(2\sigma(v) + \frac{\pi}{3}\right)\right) \\ &> 2\varepsilon \cdot \left(\sqrt{3} - \sin\left(2\sigma(v) + \frac{\pi}{3}\right)\right)\end{aligned}$$

By multiplying the left side by $\frac{1}{2}$ and considering it as the sum of the product of

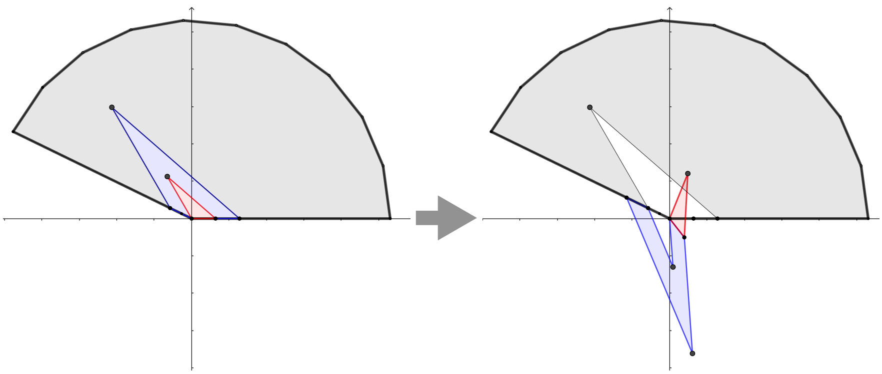

Figure 11: *Transformation of the unfolding diagram*

trigonometric functions, the transformation can be made accordingly.

$$
\begin{aligned}
& s\left(\frac{1}{2}\sin\left(2\sigma(v)+\frac{\pi}{3}\right)+\frac{\sqrt{3}}{2}\cos\left(2\sigma(v)+\frac{\pi}{3}\right)\right) \\
& = s\left(\cos\frac{\pi}{3}\sin\left(2\sigma(v)+\frac{\pi}{3}\right)+\sin\frac{\pi}{3}\cos\left(2\sigma(v)+\frac{\pi}{3}\right)\right) \\
& = s\cdot\sin\left(2\sigma(v)+\frac{2\pi}{3}\right)
\end{aligned}
$$

Therefore, the inequality can be transformed into:

$$s\cdot\sin\left(2\sigma(v)+\frac{2\pi}{3}\right) > \varepsilon\cdot\left(\sqrt{3}-\sin\left(2\sigma(v)+\frac{\pi}{3}\right)\right)$$

Given that $\frac{2\pi}{3} < \sigma(v) < \pi$, it follows that $\sin\left(2\sigma(v)+\frac{2\pi}{3}\right) > 0$, and the equation can be transformed into:

$$s > \varepsilon\frac{\sqrt{3}-\sin\left(2\sigma(v)+\frac{\pi}{3}\right)}{\sin\left(2\sigma(v)+\frac{2\pi}{3}\right)} \tag{2}$$

Thus, for any $\sigma(v)$ satisfying $\frac{2\pi}{3} < \sigma(v) < \pi$, by choosing s such that point D fits within F and taking ε sufficiently small, an unfolding with overlaps can be obtained.

Case where v satisfies condition C: In this case, a sector F with a central angle greater than or equal to $\frac{\pi}{2}$ and less than $\frac{2\pi}{3}$ is obtained (Figure 12).

Here, take circles C_1, C_2, and C_3 centered at the origin O with radii ε, 2ε, and 3ε, respectively. Let the intersection points of each C_i with the radius of F be A_i, A_i'. Also, place a point B_1 on the y-axis at a distance s from the origin. Furthermore, draw lines l_1 and l_2 parallel to the y-axis through A_1' and A_2', respectively, and lines l_3 and l_4 parallel to line A_1B_1 through A_2 and A_3, respectively. Denote their intersection points as shown in Figure 13 as B_1, B_2, B_3, D_1, D_2. □

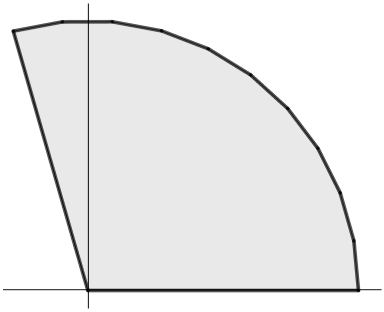

Figure 12: *Unfolding method for a vertex satisfying condition (C)*

Similar transformations, as mentioned before, result in a partial unfolding of Q, F' (Figure 14). At this point, denote the point corresponding to B_1 in the glued triangle OB_1A_1 as X. The coordinates of X become $\left(s\cos\left(3\sigma(v)+\frac{\pi}{2}\right), s\sin\left(3\sigma(v)+\frac{\pi}{2}\right)\right)$.

To demonstrate that the unfolding has overlaps, it suffices to show that there exist values of s, ε such that X lies above the line A_3D_2. The equation of line A_3D_2 can be expressed as:

$$y = s\left(\frac{-1}{\varepsilon}x + 3\right)$$

The condition for point X to be above this line is:

$$s \cdot \cos\left(3\sigma(v)+\frac{\pi}{2}\right) > s\left(\frac{-1}{\varepsilon}\left(s \cdot \cos\left(3\sigma(v)+\frac{\pi}{2}\right)\right)+3\right)$$

This inequality can be transformed, given that both s and ε are positive, to:

$$\varepsilon \cdot \cos\left(3\sigma(v)+\frac{\pi}{2}\right) > -s \cdot \cos\left(3\sigma(v)+\frac{\pi}{2}\right)+3\varepsilon$$

$$s \cdot \cos\left(3\sigma(v)+\frac{\pi}{2}\right) > -\varepsilon \cdot \cos\left(3\sigma(v)+\frac{\pi}{2}\right)+3\varepsilon$$

Given that $\frac{\pi}{2} < \sigma(v) < \frac{2\pi}{3}$, the equation can be transformed to:

$$s > -\varepsilon\frac{\cos\left(3\sigma(v)+\frac{\pi}{2}\right)+3}{\cos\left(3\sigma(v)+\frac{\pi}{2}\right)} \tag{3}$$

Therefore, for any $\sigma(v)$ satisfying $\frac{\pi}{2} < \sigma(v) < \frac{2\pi}{3}$, by choosing s such that point D_2 fits within F and taking ε sufficiently small, an unfolding with overlaps can be obtained.

4 Conclusion

In this study, we defined the concept of overlap-free and provided its characterization. The concept can be extended to what is called edge-overlap-free, meaning

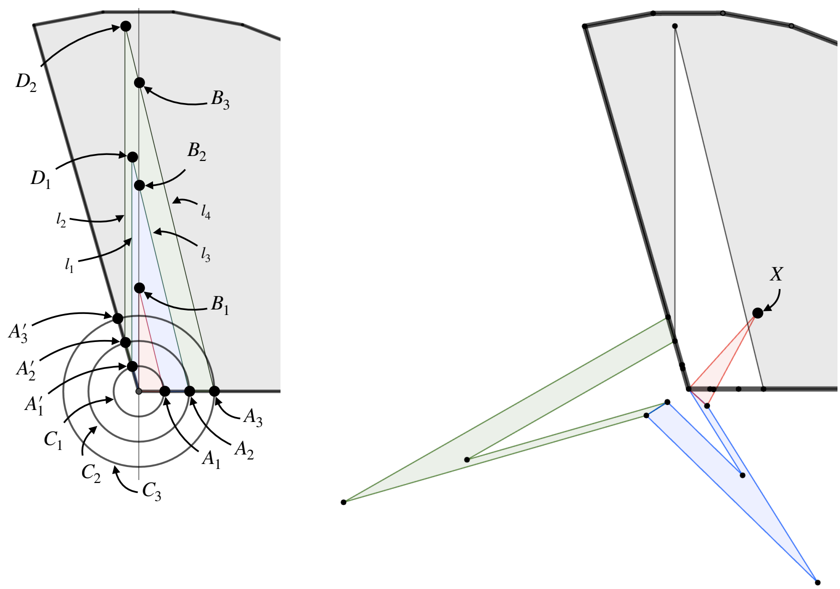

Figure 13: *Design of the cut line*

”any edge unfolding does not have overlaps.” For example, it is known that all five Platonic solids are edge-overlap-free [Horiyama and Shoji 11]. The problem of characterizing polyhedra that are edge-overlap-free remains an intriguing open question.

References

[Akiyama and Matsunaga 18] Jin Akiyama and Kiyoko Matsunaga. “Only Isotetrahedra Can Be Stampers.” *Mathematics Magazine* 91:3 (2018), 187–191. doi:10.1080/0025570X.2018.1447718. Available online (https://doi.org/10.1080/0025570X.2018.1447718).

[Akiyama and Matsunaga 20] Jin Akiyama and Kiyoko Matsunaga. “An Algorithm for Folding a Conway Tile into an Isotetrahedron or a Rectangle Dihedron.” *Journal of Information Processing* 28 (2020), 750–758. doi:10.2197/ipsjjip.28.750.

[Akiyama and Nakamura 00] Jin Akiyama and G Nakamura. “Variations of tiling problems.” *Teaching Math. Appl.* 19. doi:10.1093/teamat/19.1.8.

[Aronov and O’Rourke 92] Boris Aronov and Joseph O’Rourke. “Nonoverlap of the Star Unfolding.” *Discret. Comput. Geom.* 8 (1992), 219–250. doi:10.1007/BF02293047. Available online (https://doi.org/10.1007/BF02293047).

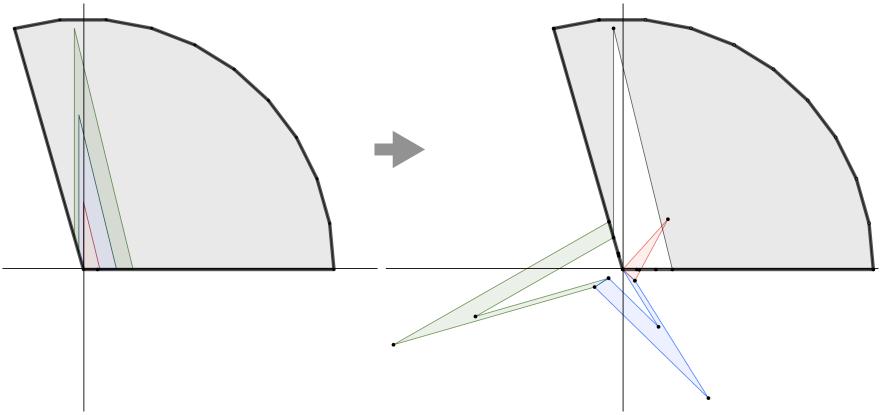

Figure 14: *Transformation of the unfolding diagram*

[Horiyama and Shoji 11] Takashi Horiyama and Wataru Shoji. "Edge Unfoldings of Platonic Solids Never Overlap." In *23rd Canadian Conference on Computational Geometry*, 2011.

[Kamata et al. 22] Tonan Kamata, Akira Kadoguchi, Takashi Horiyama, and Ryuhei Uehara. "Efficient Folding Algorithms for Convex Polyhedra." *Discrete & Computational Geometry (accepted).*

[Namiki and Fukuda 93] Makoto Namiki and Komei Fukuda. "Unfolding 3-dimensional convex polytopes." *A package for Mathematica 1.2 or 2.0. Mathematica Notebook.*

[Sharir and Schorr 86] Micha Sharir and Amir Schorr. "On Shortest Paths in Polyhedral Spaces." *SIAM J. Comput.* 15:1 (1986), 193–215. doi:10.1137/0215014. Available online (https://doi.org/10.1137/0215014).

[Shephard 75] G. C. Shephard. "Convex polytopes with convex nets." *Mathematical Proceedings of the Cambridge Philosophical Society* 78:3 (1975), 389–403. doi:10.1017/S0305004100051860.

Tonan Kamata
Japan Advanced Institute of Science and Technology, Japan, e-mail: Kamata@jaist.ac.jp

Takumi Shiota
Kyushu Institute of Technology, Japan, e-mail: shiota.takumi779@mail.kyutech.jp

Ryuhei Uehara
Japan Advanced Institute of Science and Technology, Japan, e-mail: uehara@jaist.ac.jp