A Characterization of the Overlap-free Polyhedra

Tonan Kamata (JAIST)
Takumi Shiota (Kyutech)
Ryuhei Uehara (JAIST)

The 8th International Meeting on Origami in Science, Mathematics and Education

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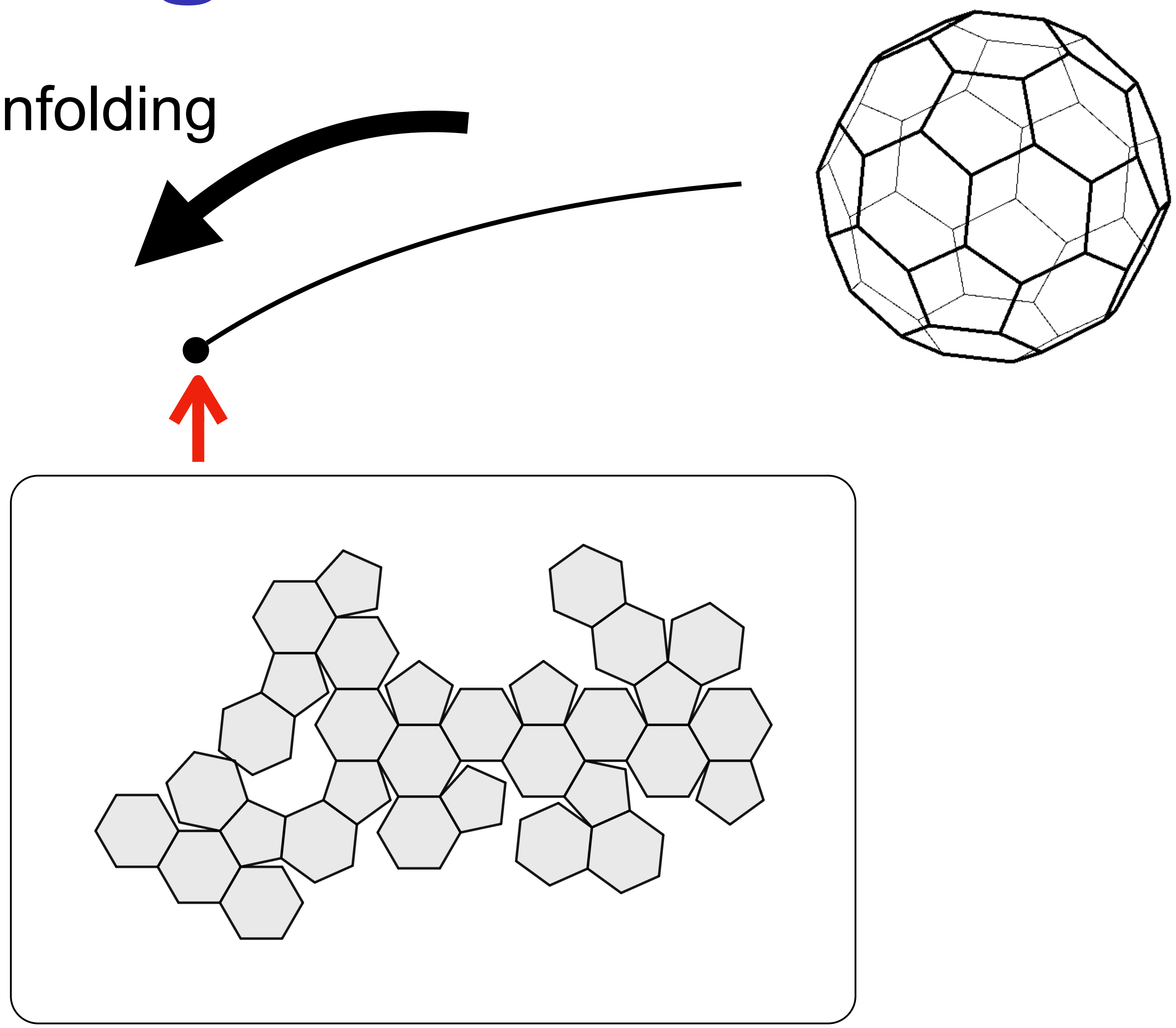
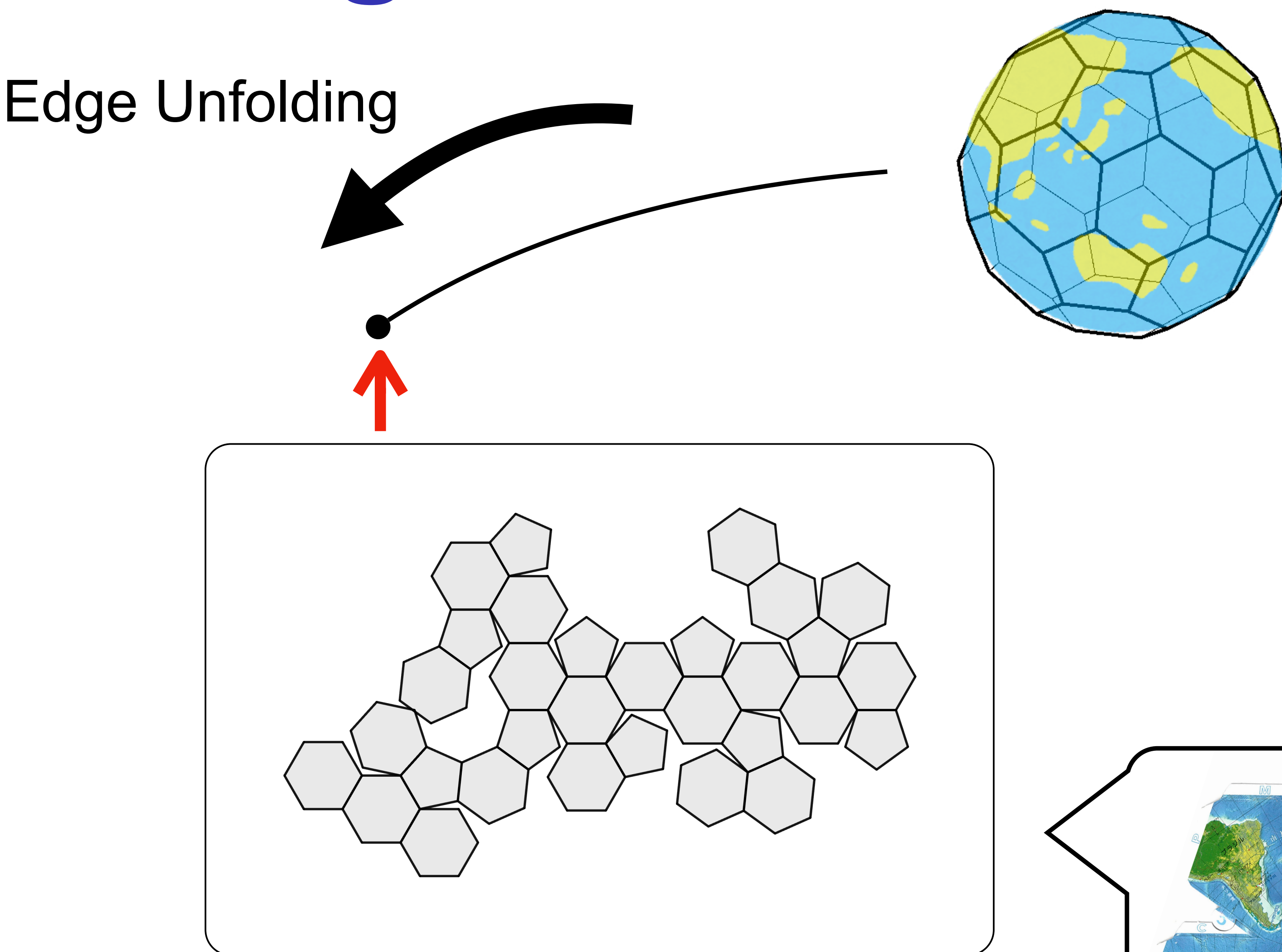
Backgrounds

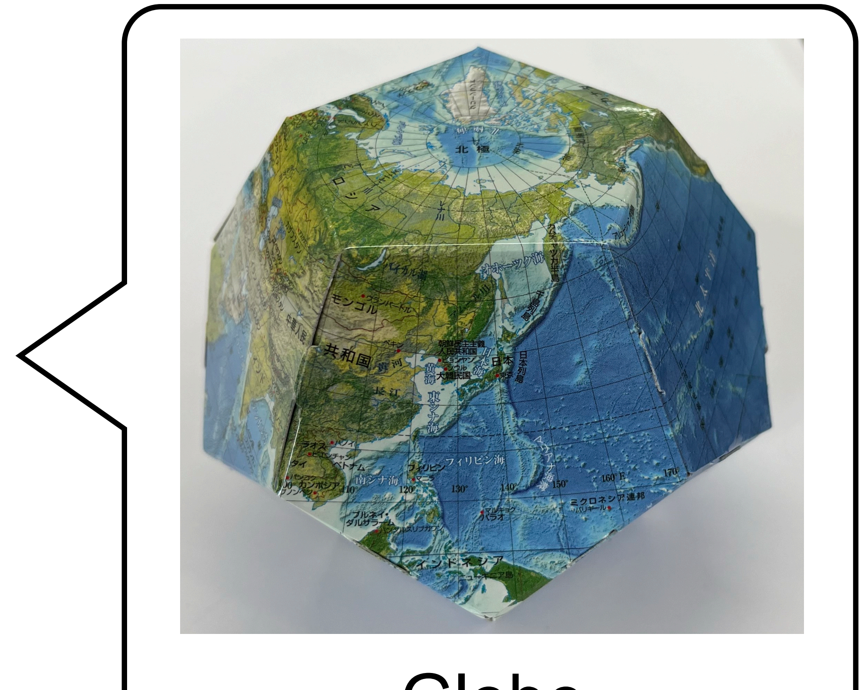
Edge Unfolding

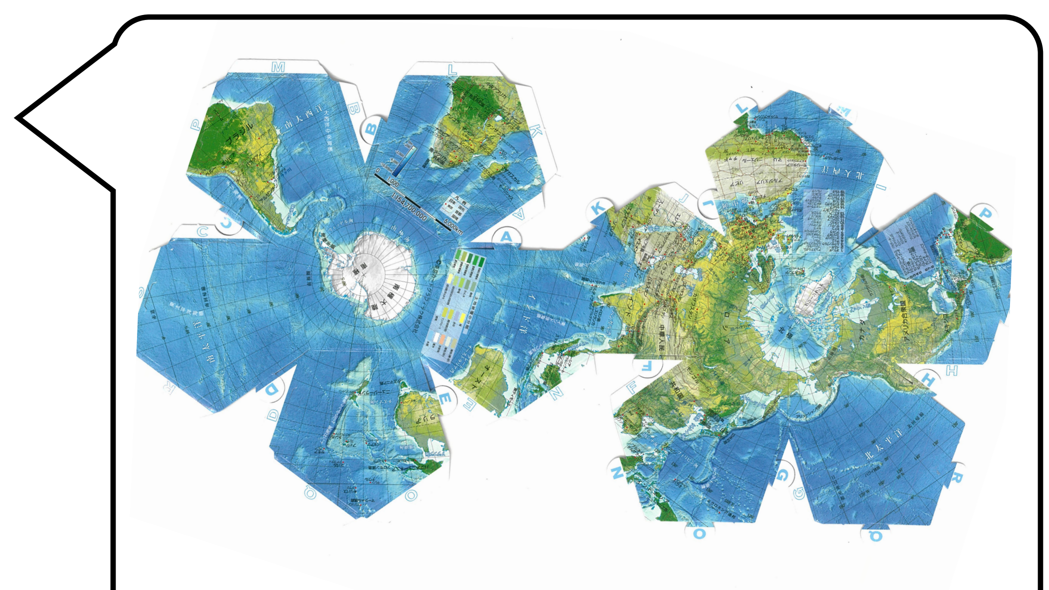


* An example from [T. Shiota and T. Saitoh, WALCOM 2023]

Backgrounds

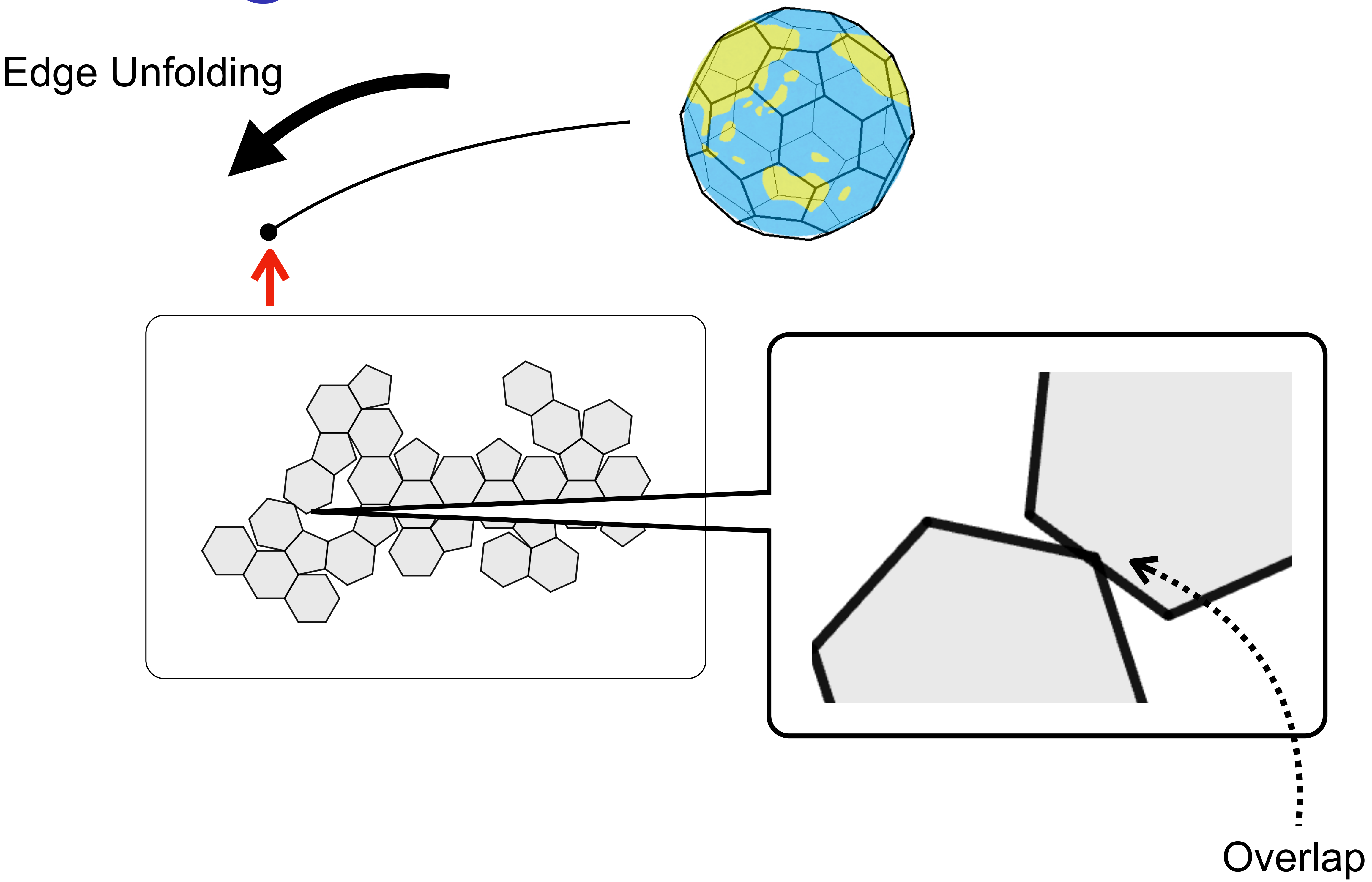



Globe


Map

* An example from [T. Shiota and T. Saitoh, WALCOM 2023]

Backgrounds



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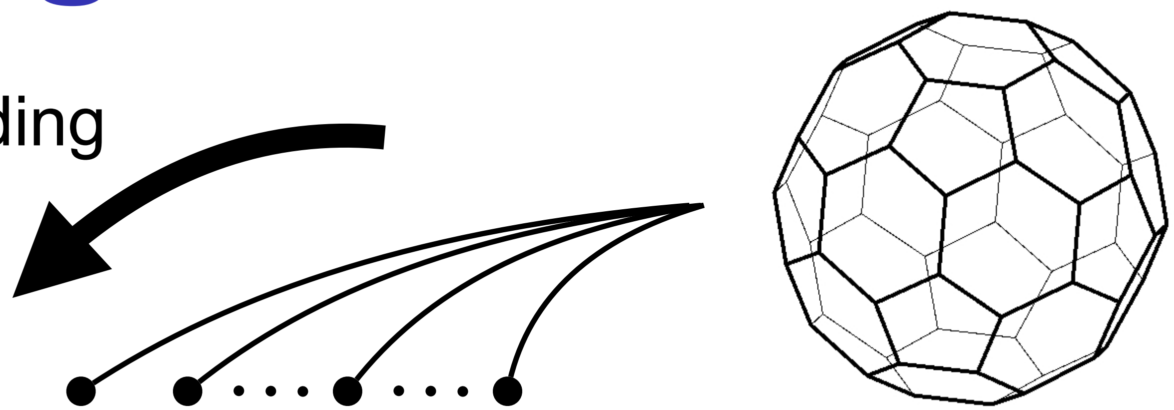

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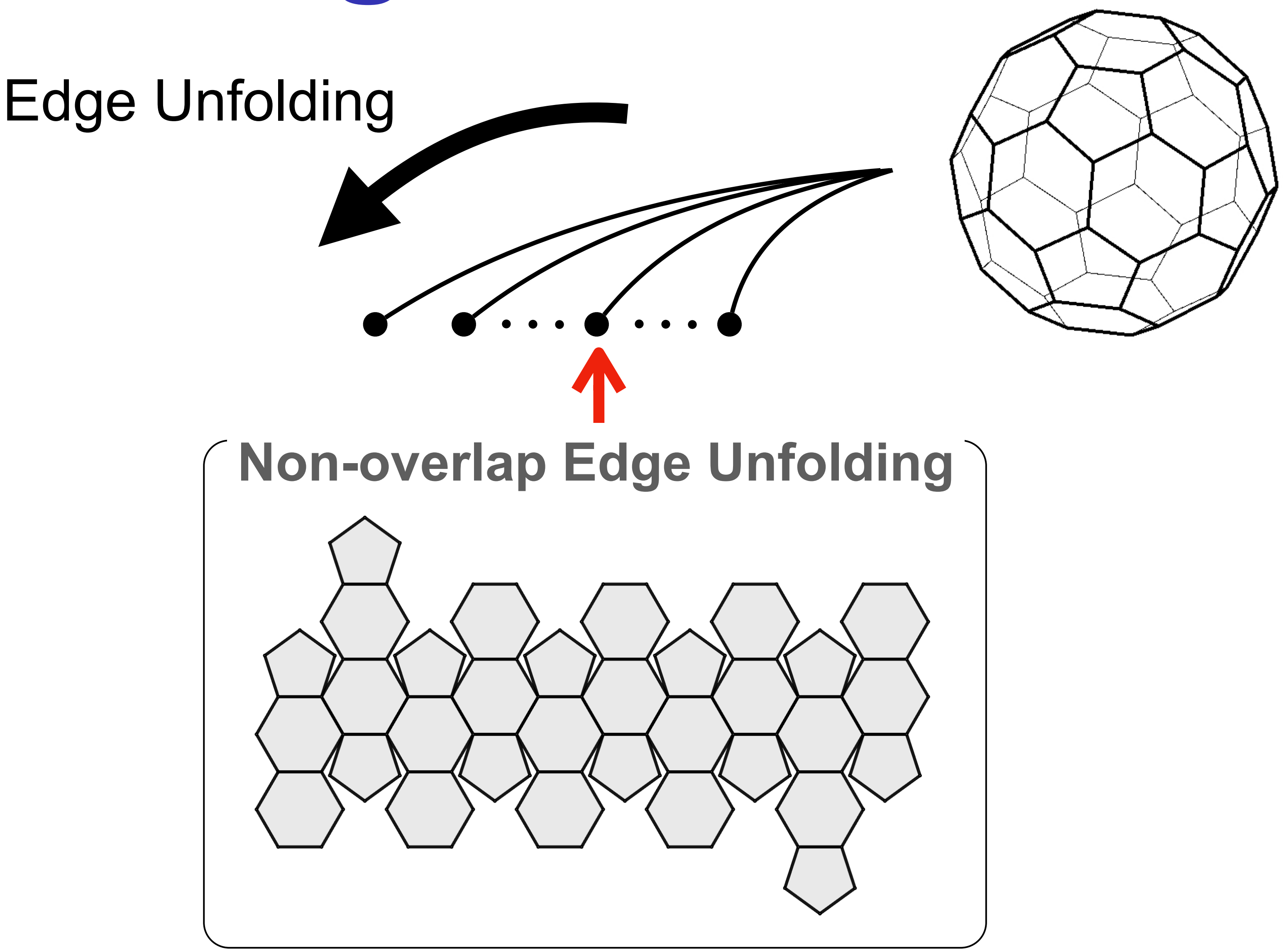

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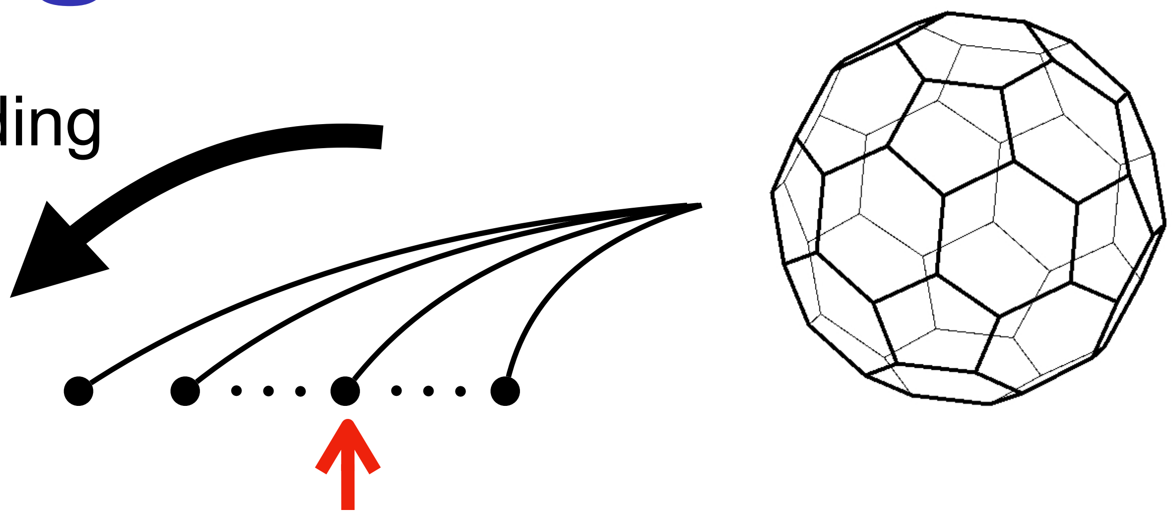
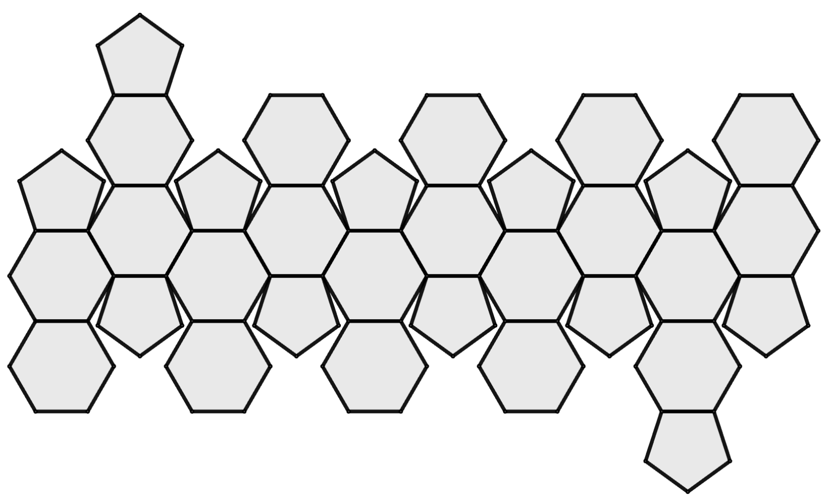


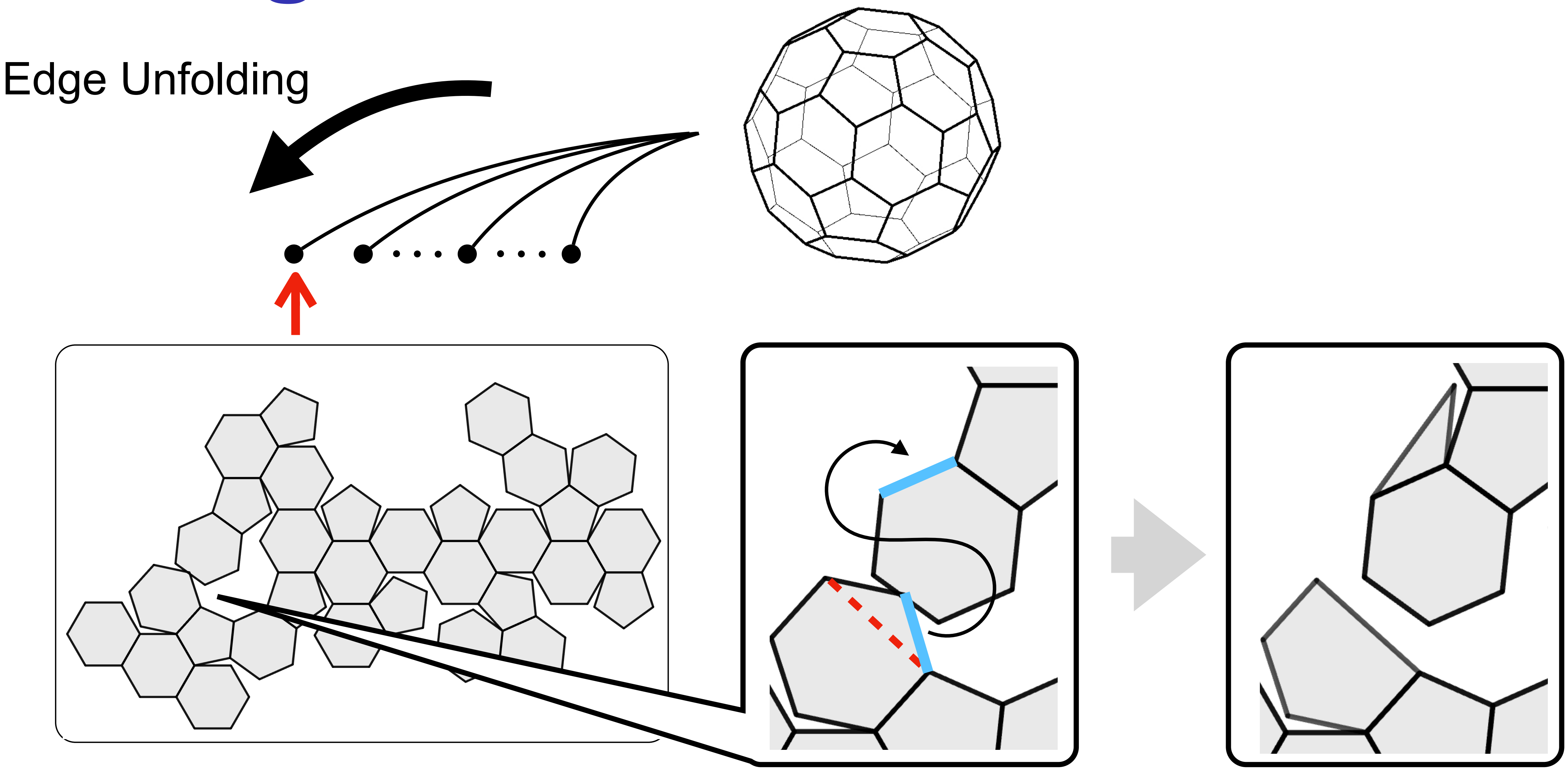
Non-overlap Edge Unfolding



Open Problem [Shephard, 1975]

Can any convex polyhedra be unfolded along edges without overlaps?

Backgrounds



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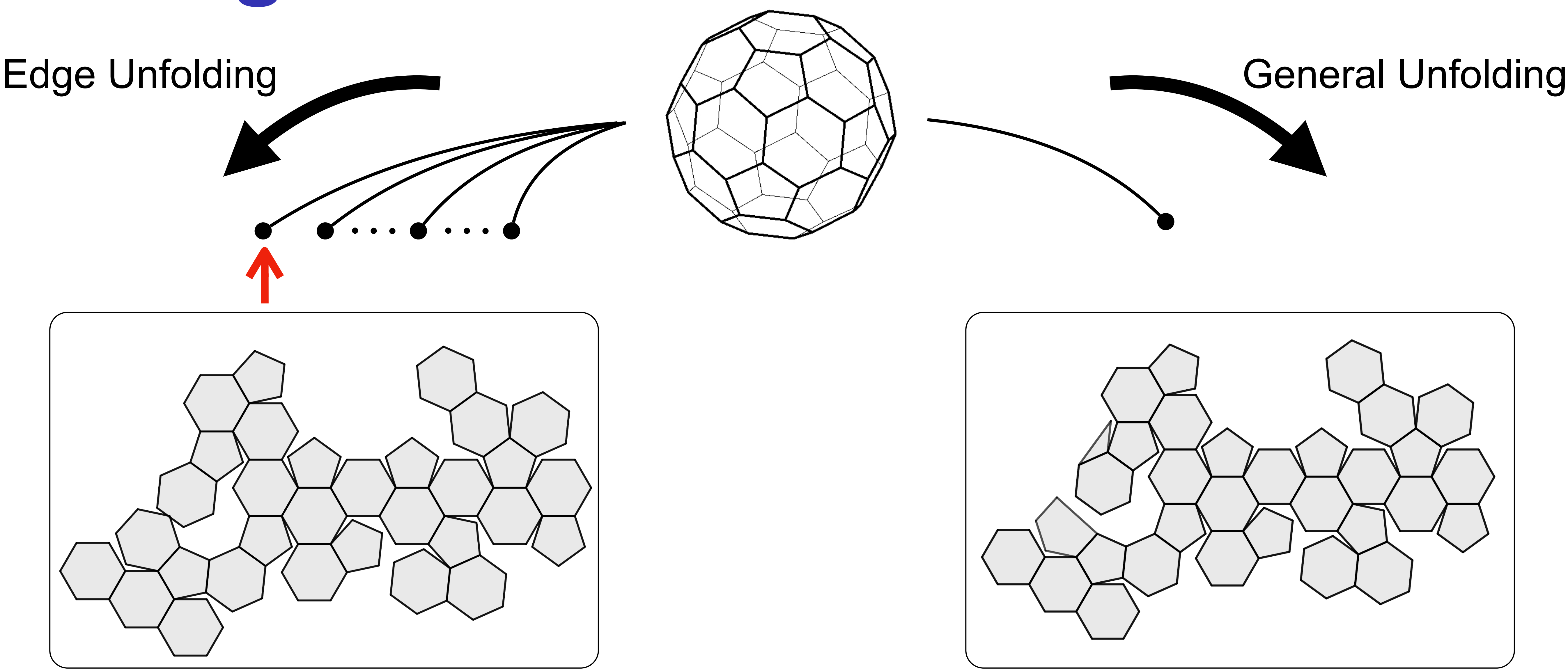

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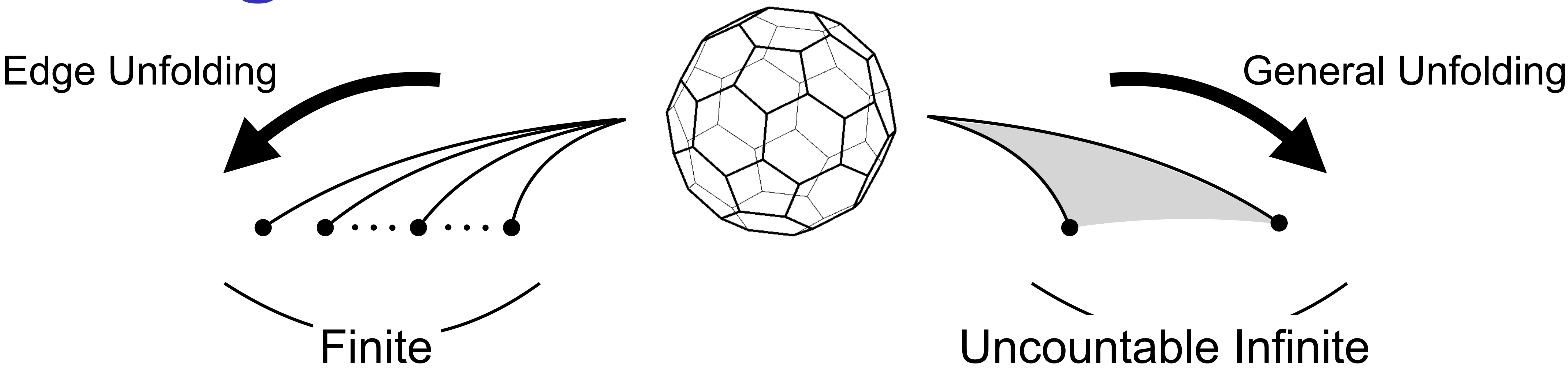

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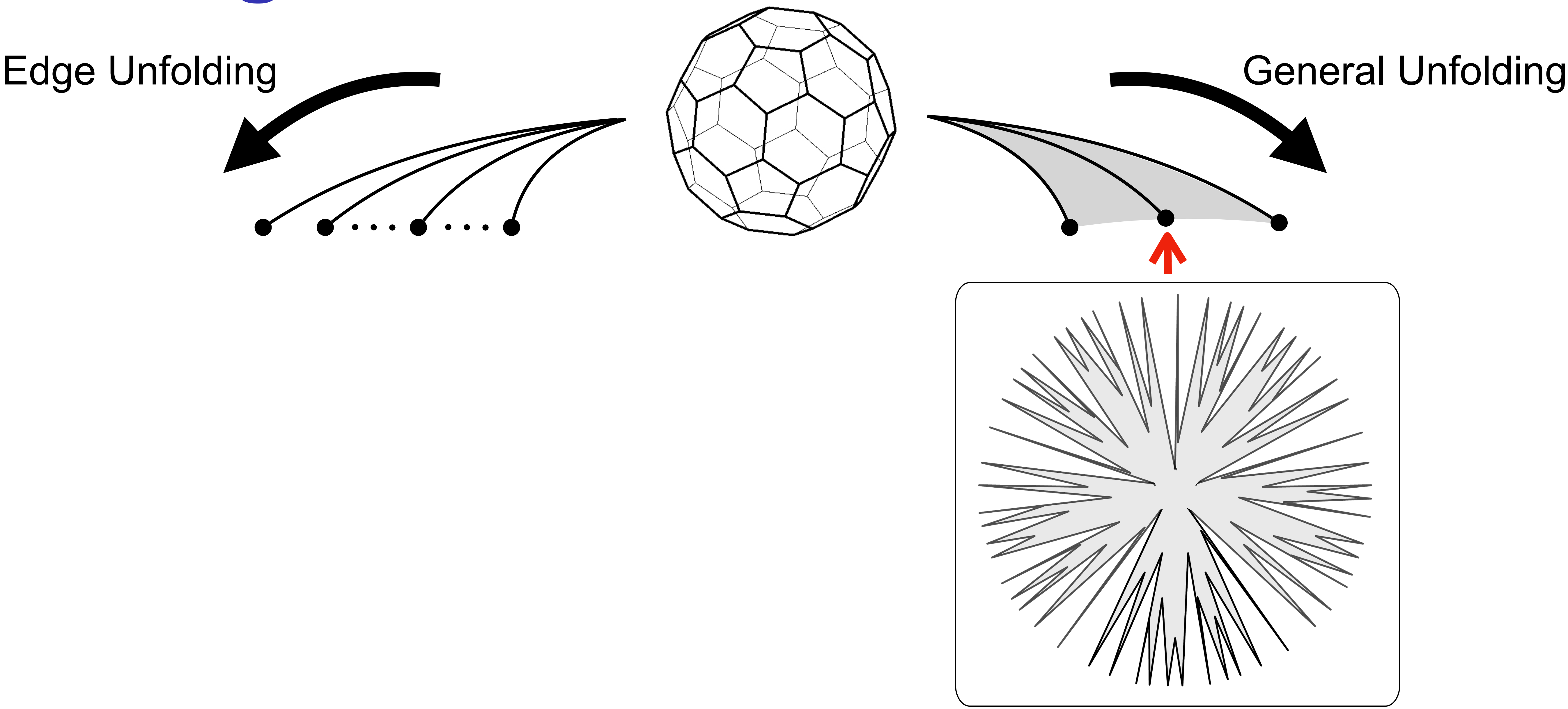

Open Problem [Shephard, 1975]	Theorem [Sharir & Schorr, 1986]
Can any convex polyhedra be unfolded along edges without overlaps?	Any convex polyhedron has a non-overlapping general unfolding.

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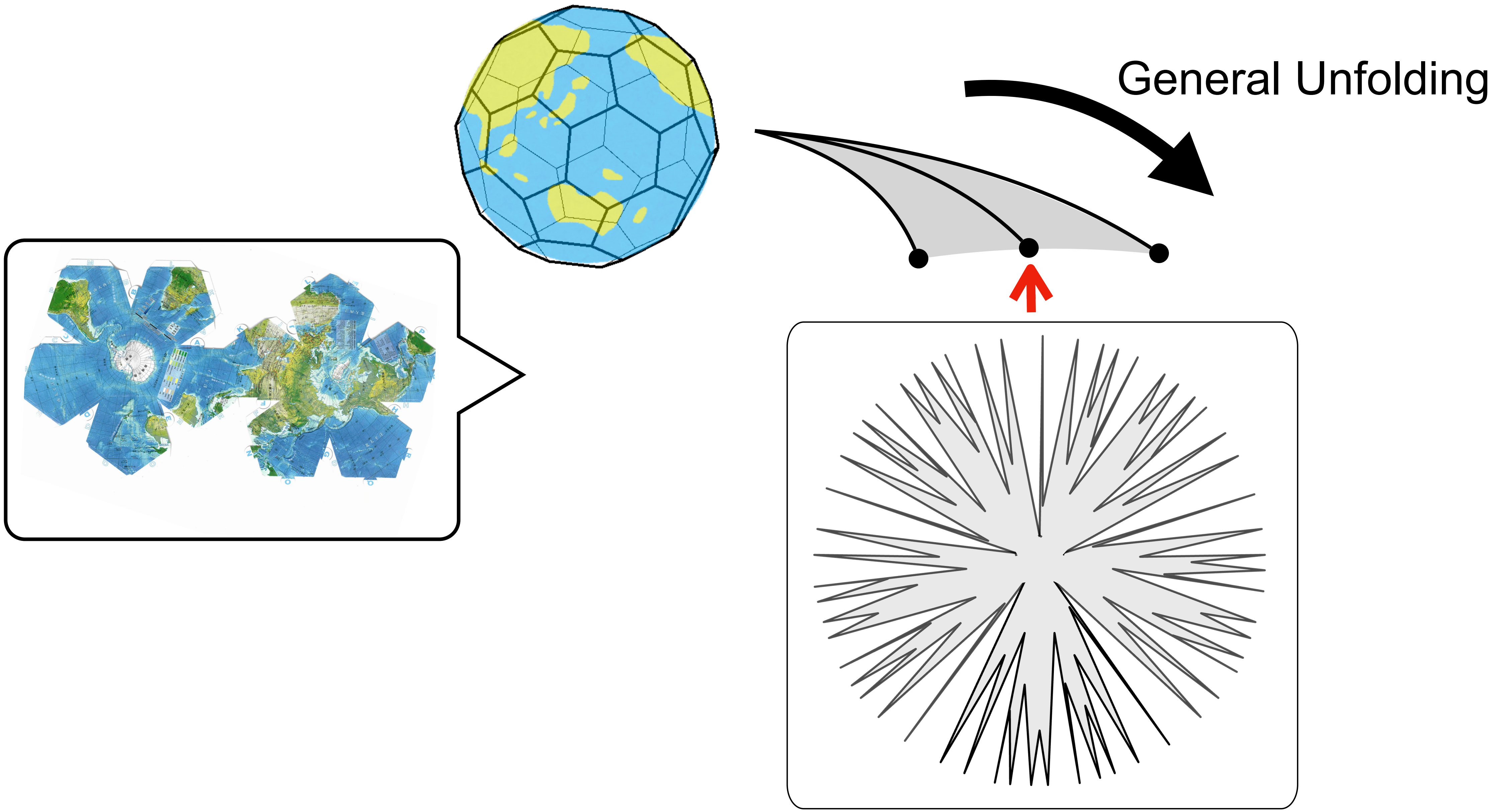
Backgrounds



Theorem [Sharir & Schorr, 1986]

Any convex polyhedron has
a non-overlapping general unfolding.

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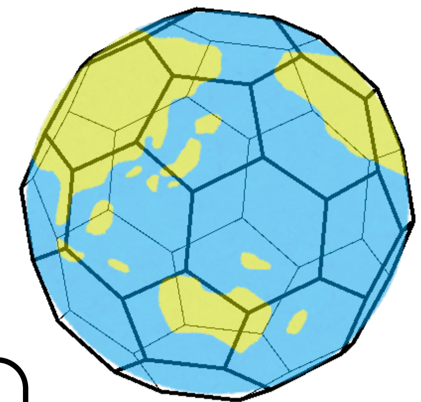
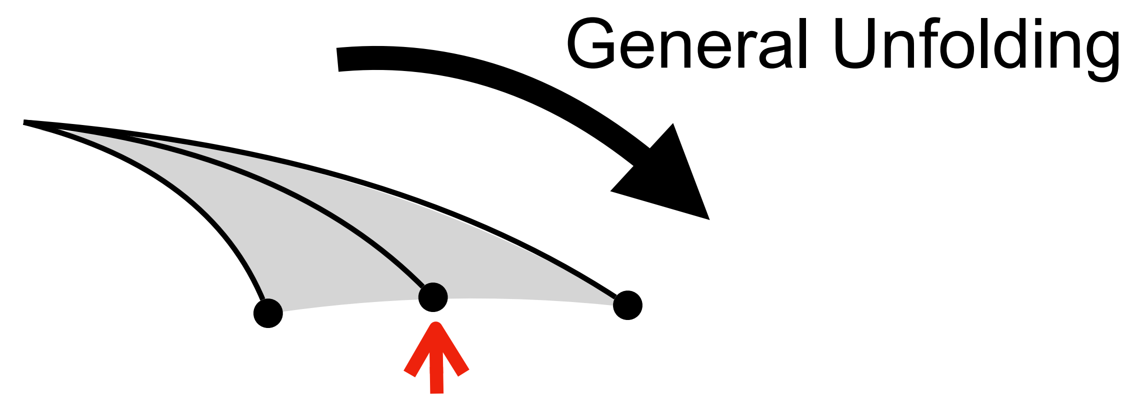
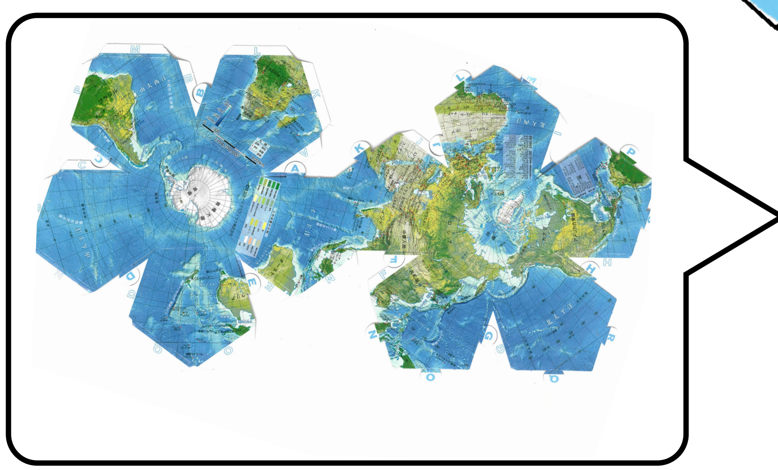
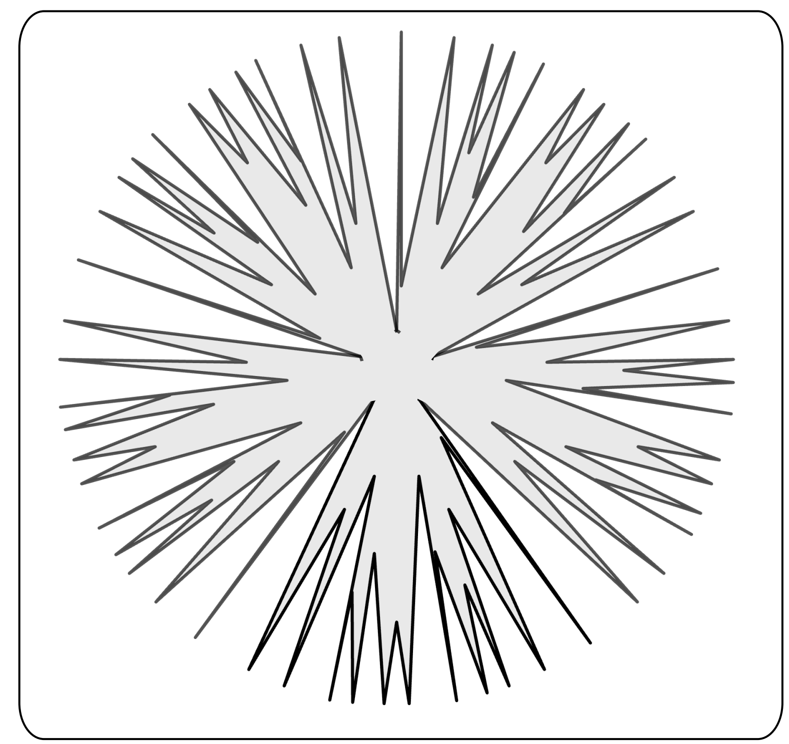
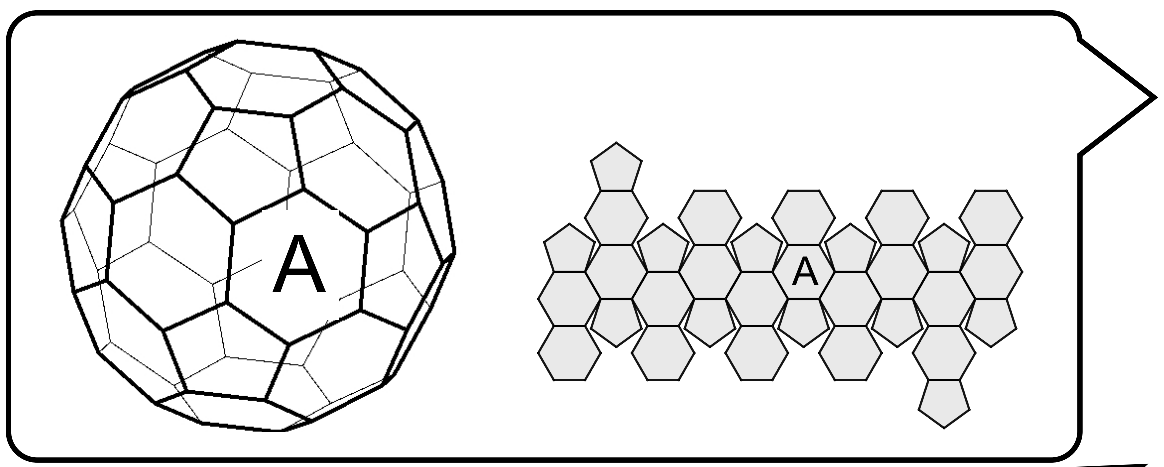
Backgrounds

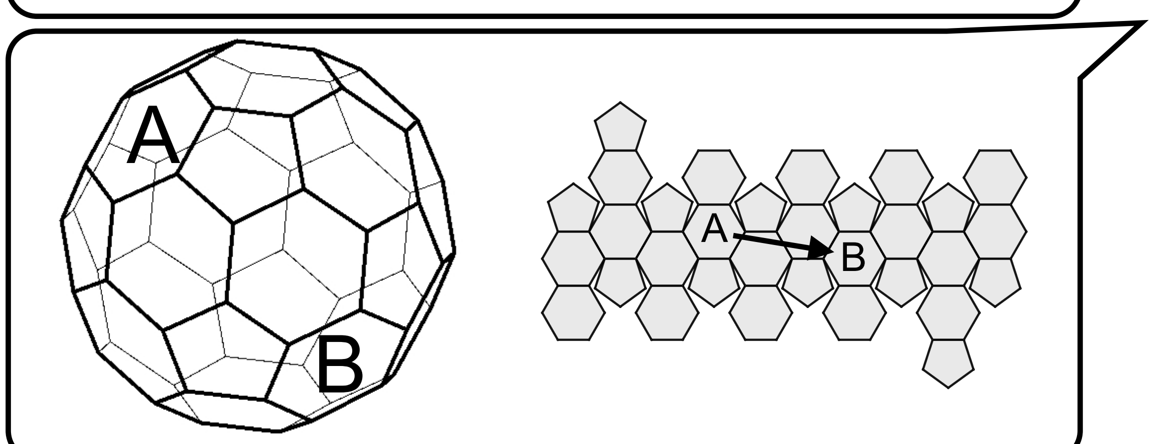












Theorem [Sharir & Schorr, 1986]

Any convex polyhedron has
a non-overlapping general unfolding.

...

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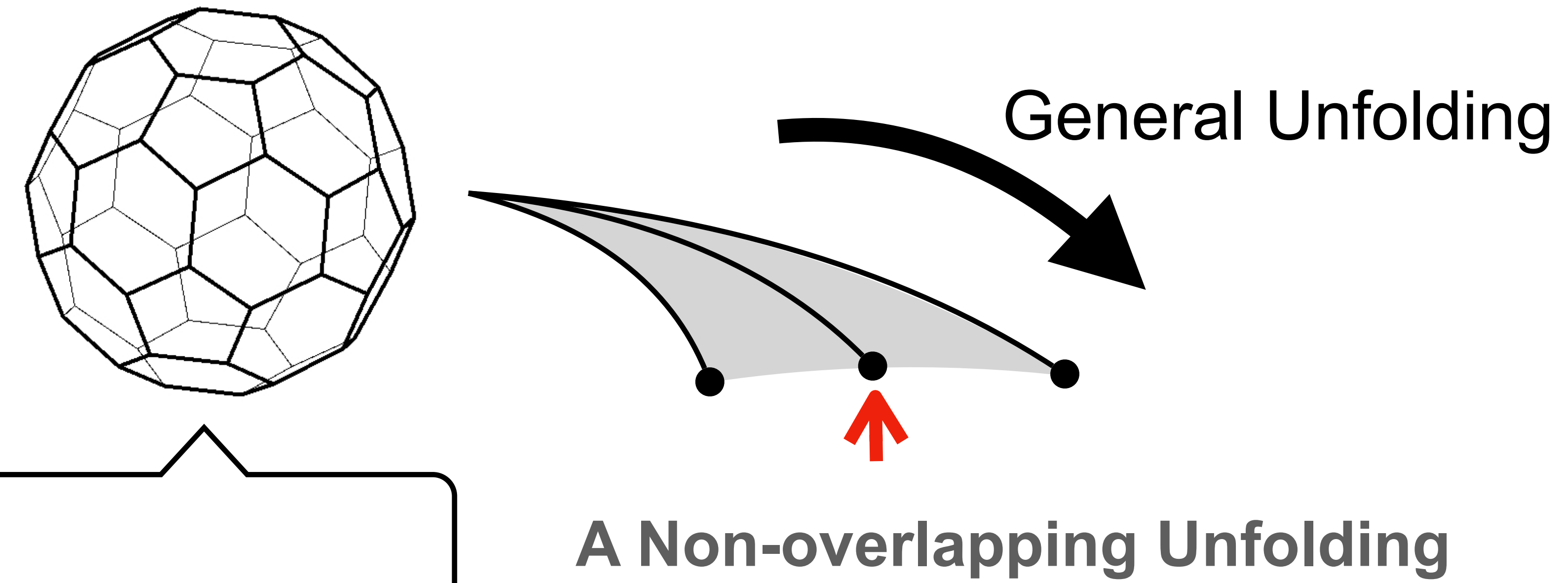
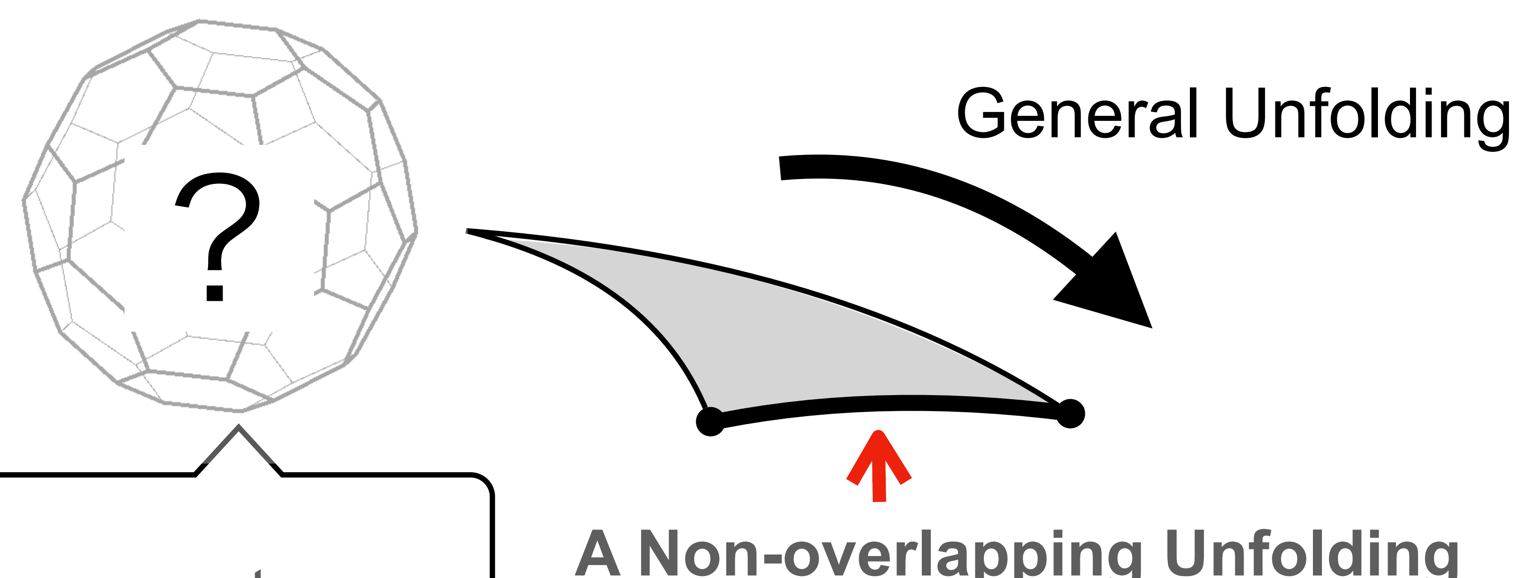


Any convex polyhedron satisfies the property
"there **exists** a non-overlapping general unfolding"



What types of polyhedra have the property
"**any** general unfolding is non-overlapping"?
(= Overlap-free)

Result

Theorem

For any convex polyhedron Q,

Q is overlap-free

$\Updownarrow$

Q is either one of

- tetramonohedron
- doubly-covered regular triangle
- doubly-covered half regular triangle
- doubly-covered right triangle

Result

Theorem

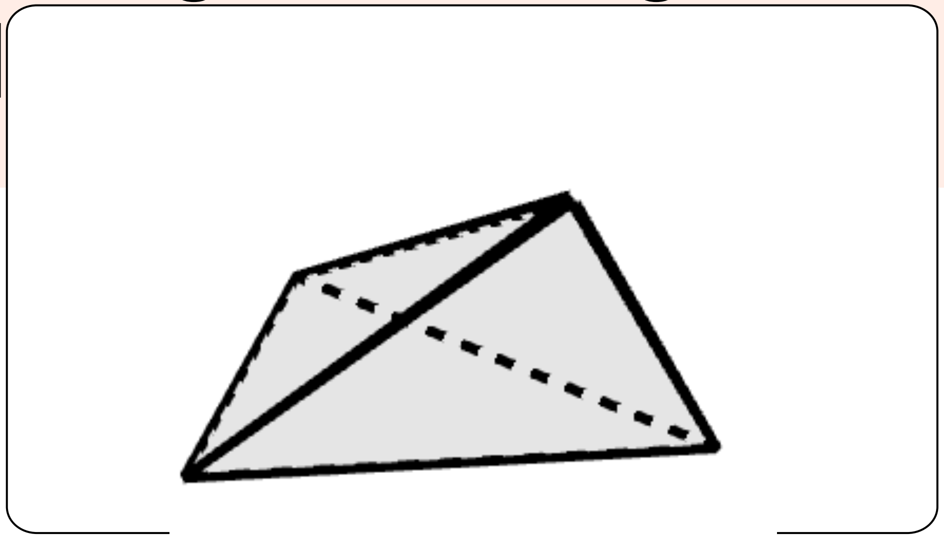
For any convex polyhedron Q,

Q is overlap-free

⇕

Q is either one of

tetramonohedron
doubly-covered regular triangle
doubly-covered half regular triangle
doubly-covered



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For any convex polyhedron Q,

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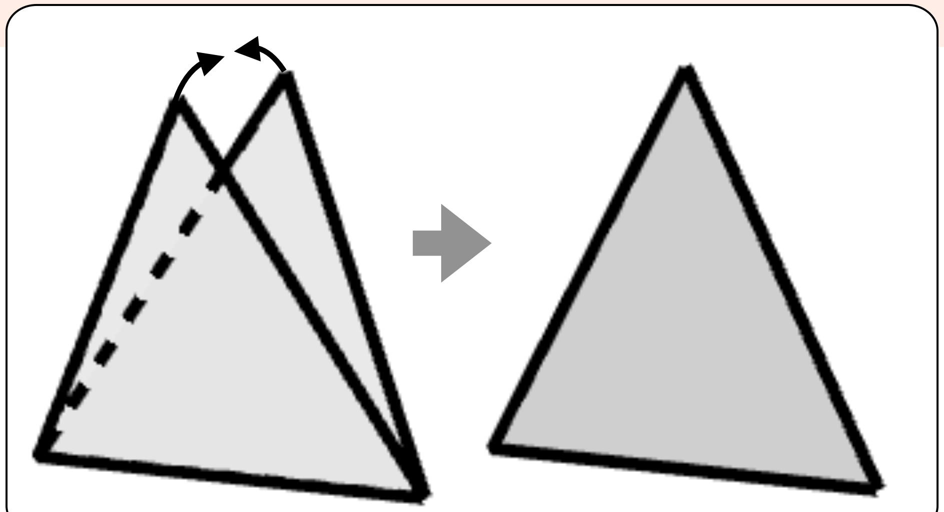
$\Updownarrow$

Q is either one of

- tetramonohedron
- **doubly-covered** regular triangle
- **doubly-covered** half regular triangle
- **doubly-covered** right triangle



Result

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Q is either one of

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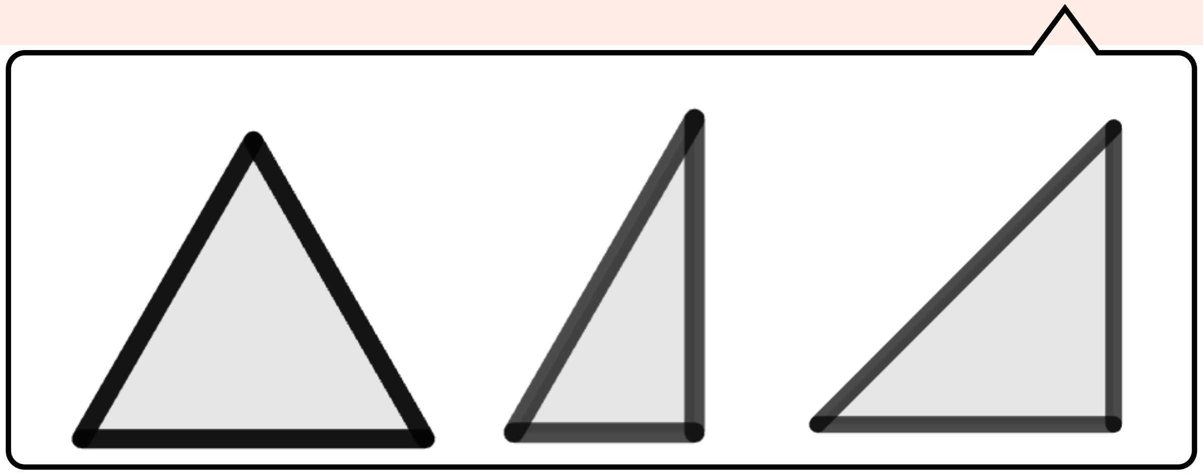


Result

Theorem

For any convex polyhedron Q,

Q is overlap-free

$\Updownarrow$

Theorem [Akiyama, 2008]

Q is either one of

- tetramonohedron
- doubly-covered **regular triangle**
- doubly-covered half **regular triangle**
- doubly-covered **right triangle**

$\Updownarrow$

Q is “stamper”

Result

Theorem

For any convex polyhedron Q,

$$Q \text{ is overlap-free} \Leftrightarrow Q \text{ is “stamper”}$$

Result

Theorem

For any convex polyhedron Q,

Q is overlap-free $\Leftrightarrow$ Q is "stamper"

Lemma

For any convex polyhedron Q,

Q is overlap-free $\Rightarrow$ Q is "stamper"

Lemma

For any convex polyhedron Q,

Q is "stamper" $\Rightarrow$ Q is overlap-free

Result

Theorem

For any convex polyhedron Q,

Q is overlap-free $\Leftrightarrow$ Q is “stamper”

Lemma

For any convex polyhedron Q,

Q is not “stamper” $\Rightarrow$ Q is not overlap-free

Result

Theorem

For any convex polyhedron Q,

Q is overlap-free $\Leftrightarrow$ Q is "stamper"

Lemma

For any convex polyhedron Q,

Q is not "stamper" $\Rightarrow$ Q is not overlap-free

Not (tetramonohedron
doubly-covered regular triangle
doubly-covered half regular triangle
doubly-covered right triangle

There exists overlap unfolding

….

Proof of Necessities - Strategy -

Lemma

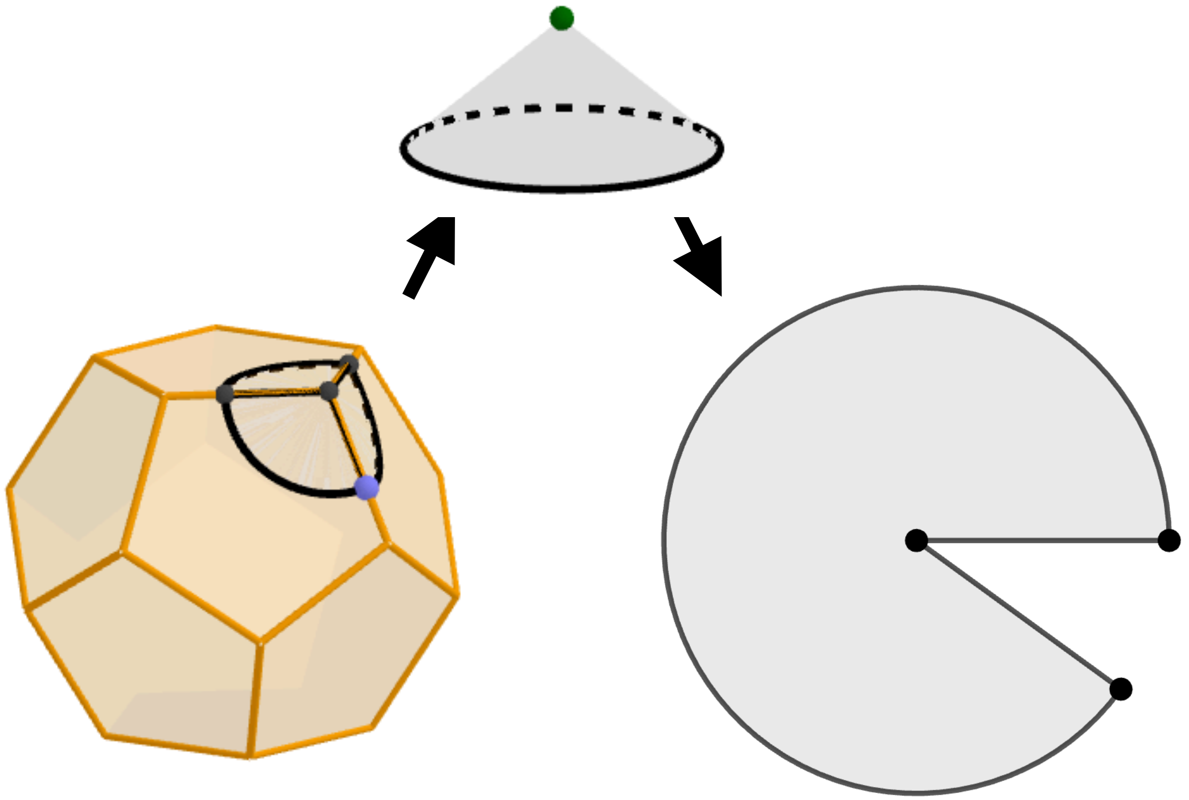
If a convex polyhedron Q is not stamper,
Q has overlapping unfolding.

Proof of Necessities - Strategy -

Lemma

If a convex polyhedron Q is not stamper,
Q has overlapping unfolding.

- Cut out a vertex to create a sector.

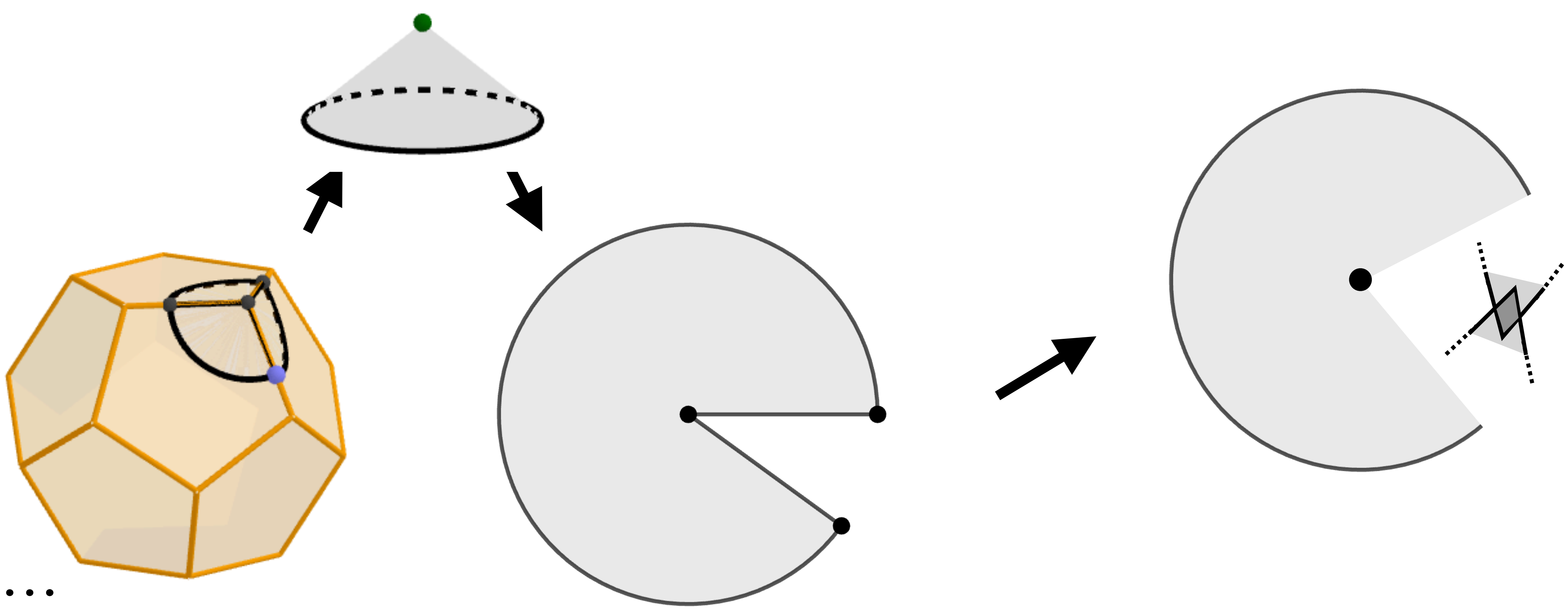


Proof of Necessities - Strategy -

Lemma

If a convex polyhedron Q is not stamper, Q has overlapping unfolding.

- Cut out a vertex to create a sector.
- Edit it to create overlaps.



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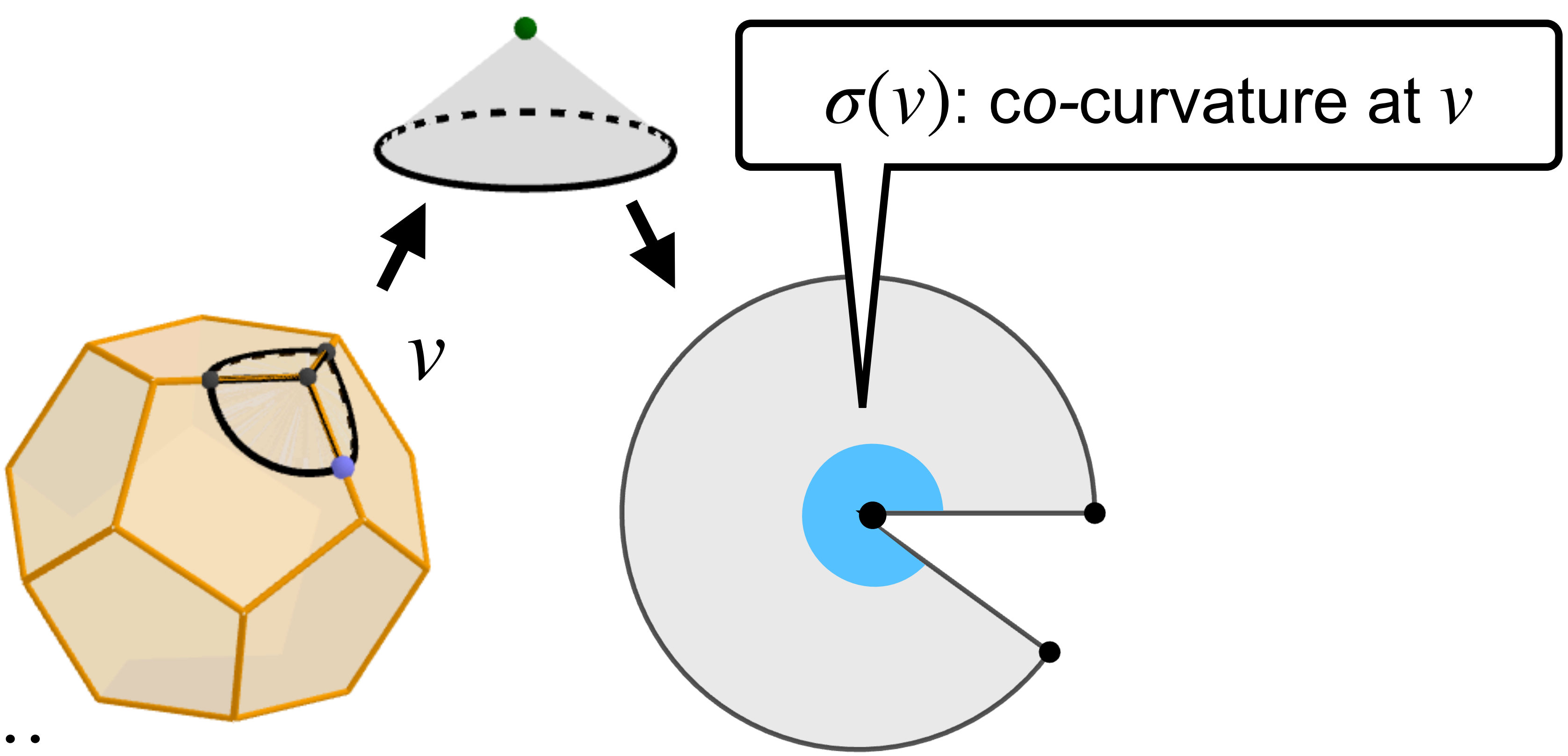

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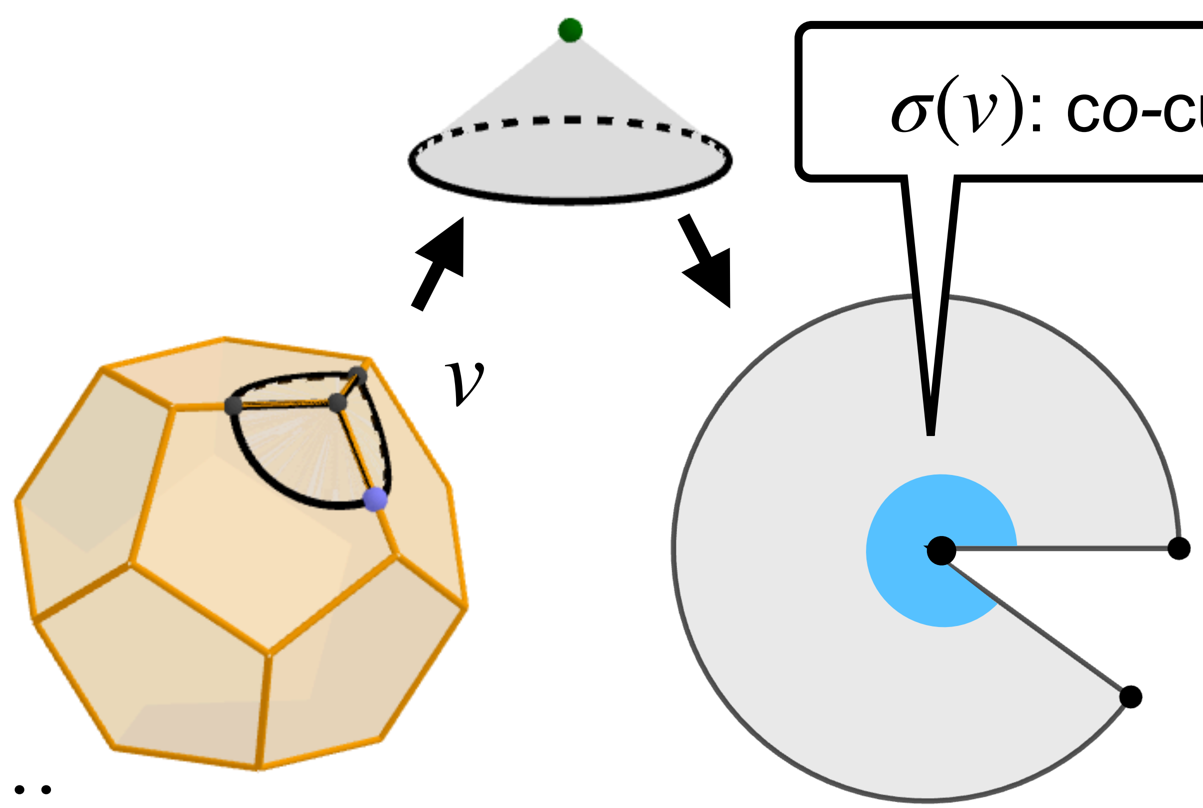

Descartes' Theorem

For a convex polyhedron Q with n vertices,

$$\sum_{v \in V(Q)} \sigma(v) = 2(n-2)\pi$$

Proof of Necessities - Strategy -

Lemma

If a convex polyhedron Q is not stamper, Q has overlapping unfolding.

n : the number of vertices of Q

$n = 3$

$n = 4$

$n > 4$

All Convex Polyhedra

Proof of Necessities - Strategy -

Lemma

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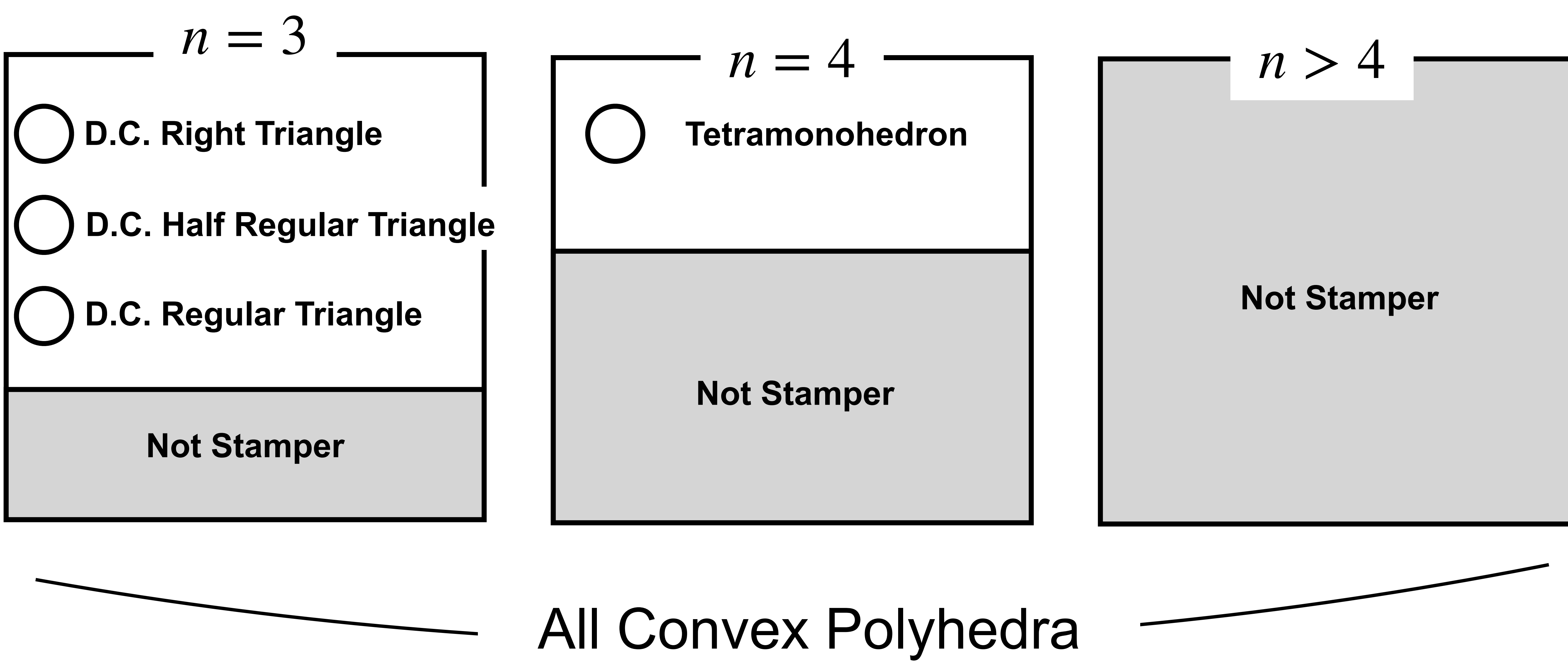
n : the number of vertices of Q



Proof of Necessities - Strategy -

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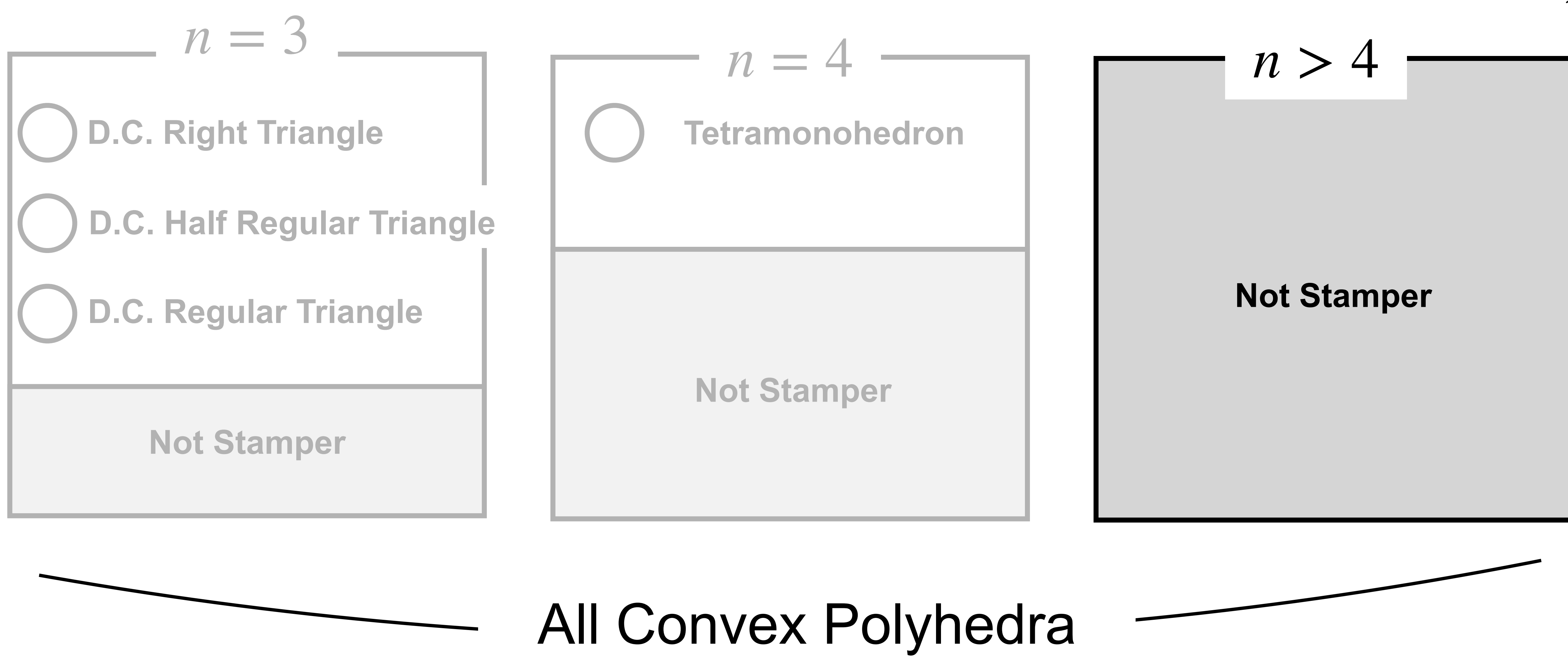
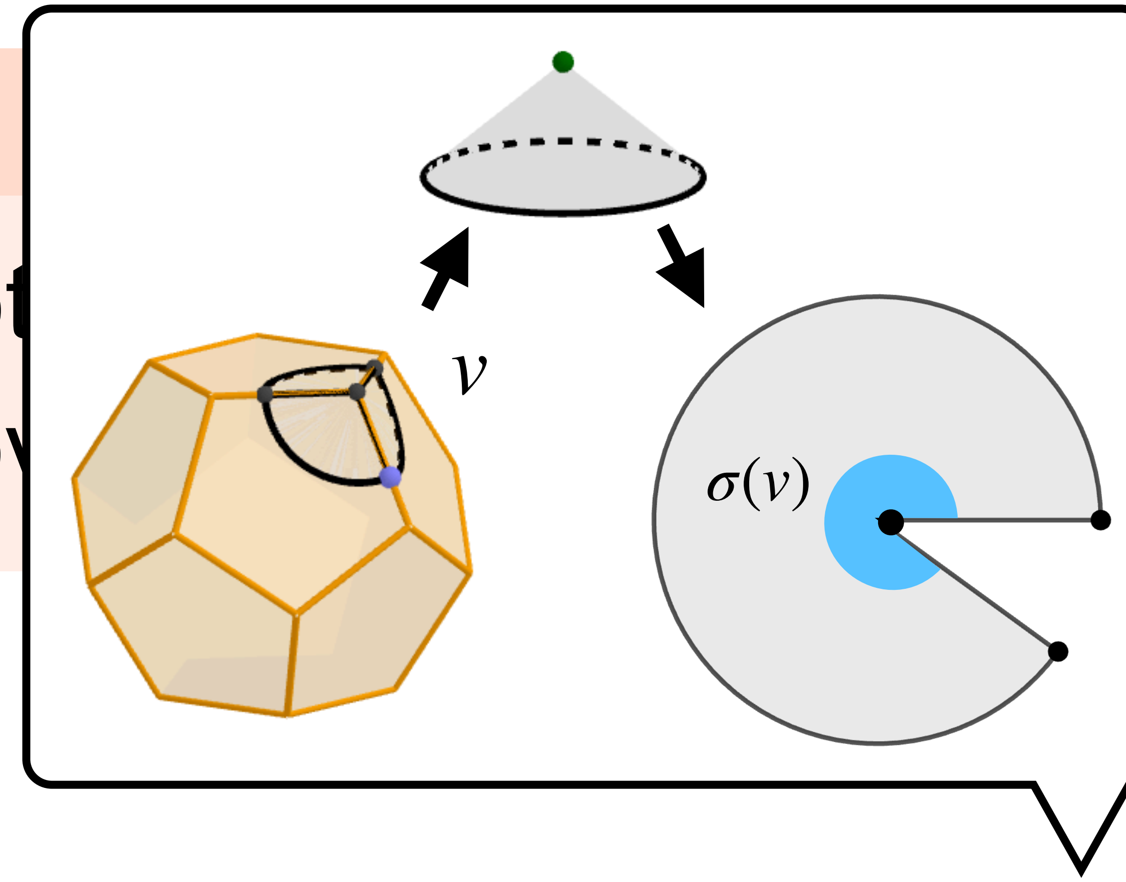


Proof of Necessities

- Details -



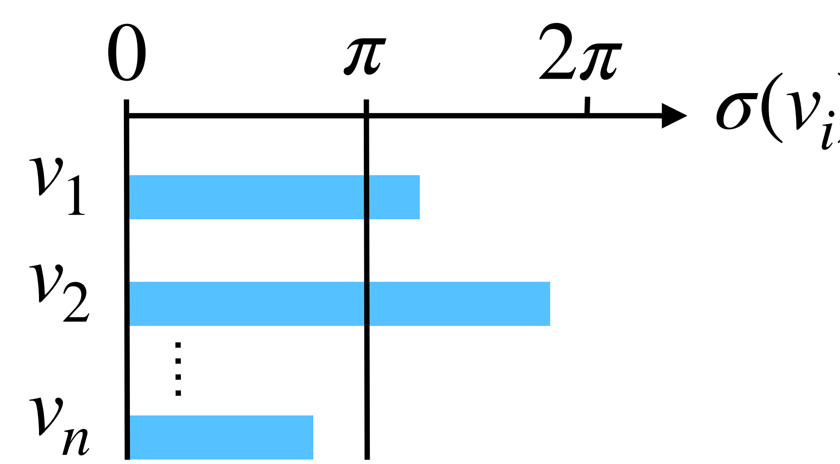
Lemma

If a convex polyhedron Q is not

Q has ov

[Proof] Case of $n > 4$

Let $v_1, v_2, \ldots, v_n$ be the vertices of Q



Proof of Necessities - Details -

Lemma

If a convex polyhedron Q is not stamper,
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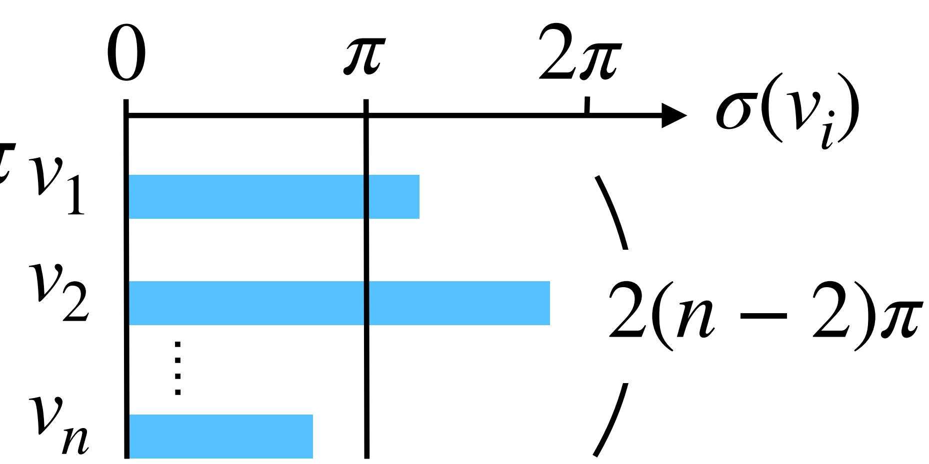
[Proof] Case of $n > 4$

Let $v_1, v_2, \ldots, v_n$ be the vertices of Q

$\Rightarrow$ From Descartes' Thm. $\sum_{v \in V(Q)} \sigma(v) = 2(n-2)\pi$



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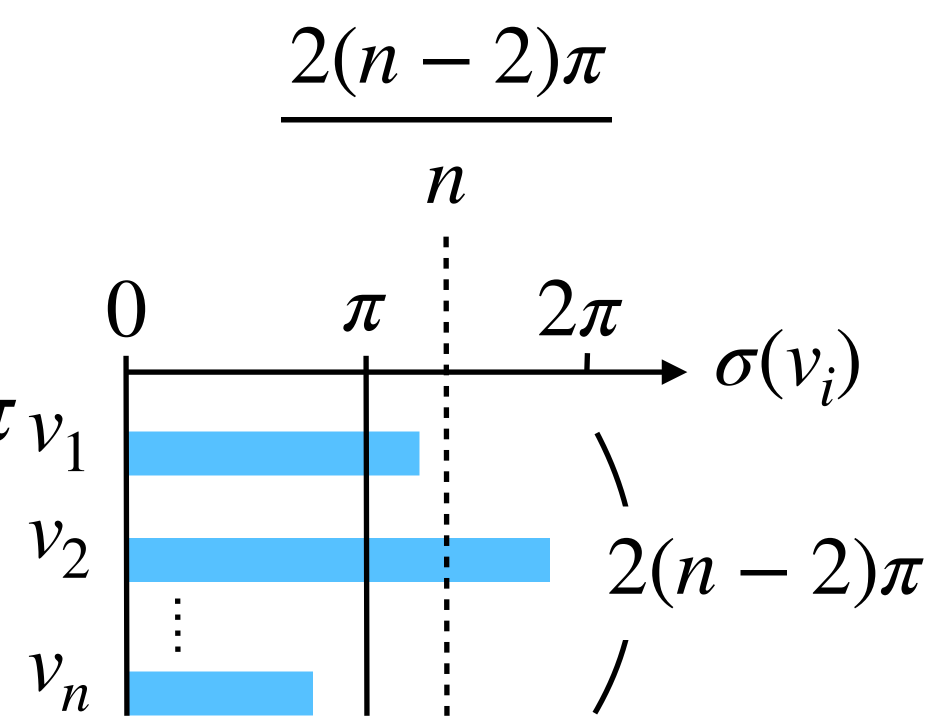
[Proof] Case of $n > 4$

Let $v_1, v_2, \ldots, v_n$ be the vertices of Q

$\Rightarrow$ From Descartes' Thm. $\sum_{v \in V(Q)} \sigma(v) = 2(n-2)\pi$

$\Rightarrow$ The average of $\sigma(v_i)$ is $\dfrac{2(n-2)\pi}{n} > \pi$



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[Proof] Case of $n > 4$

Let $v_1, v_2, \ldots, v_n$ be the vertices of Q

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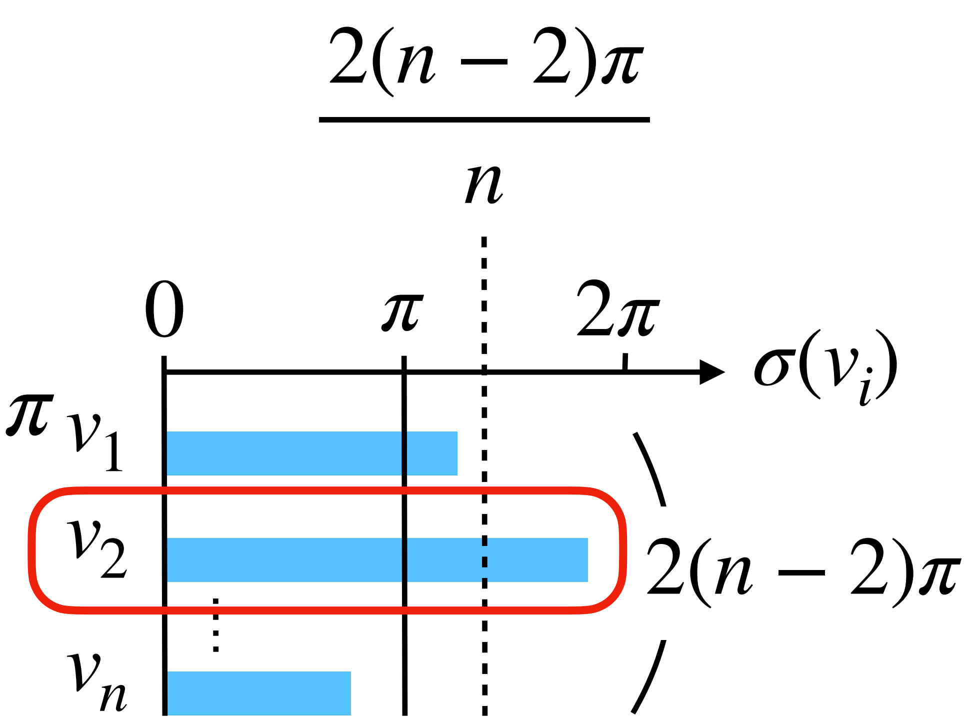
$\Rightarrow$ The average of $\sigma(v_i)$ is $\dfrac{2(n-2)\pi}{n} > \pi$

$\Rightarrow$ There is at least one v where $\sigma(v) > \pi$



Continue

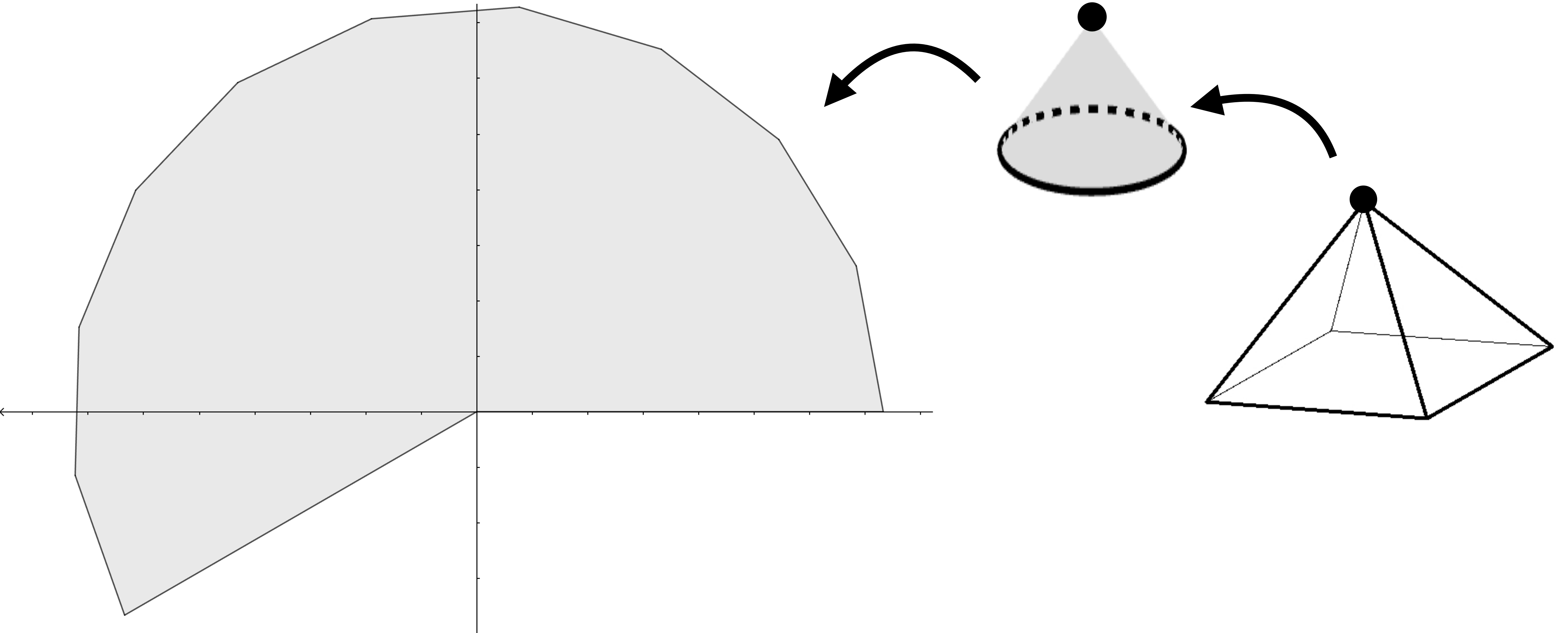
Proof of Necessities

- Details -

Lemma

If a convex polyhedron Q is not stamper, Q has overlapping unfolding.

[Proof] If $n > 4$, there is a vertex v which satisfies $\sigma(v) > \pi$



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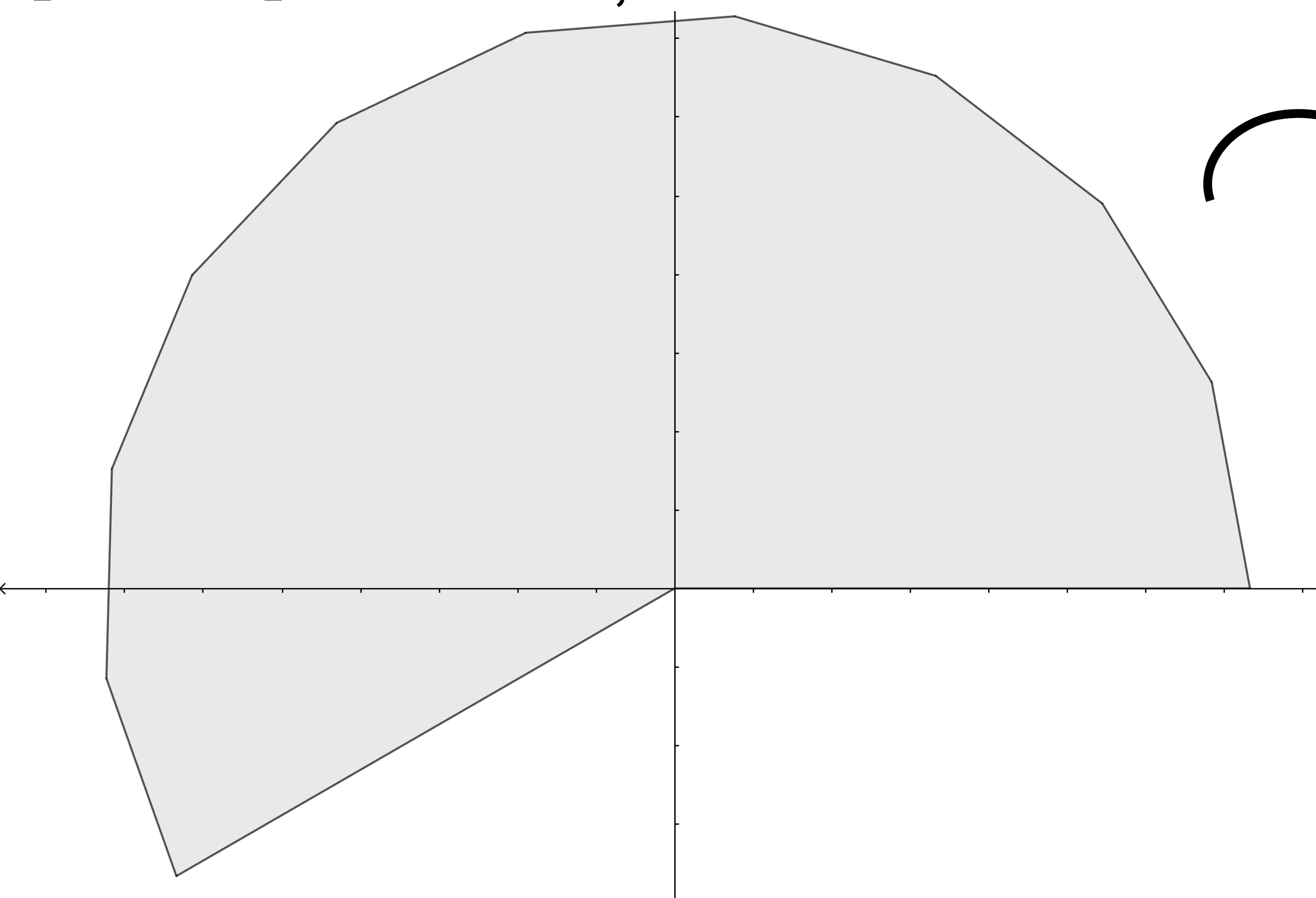

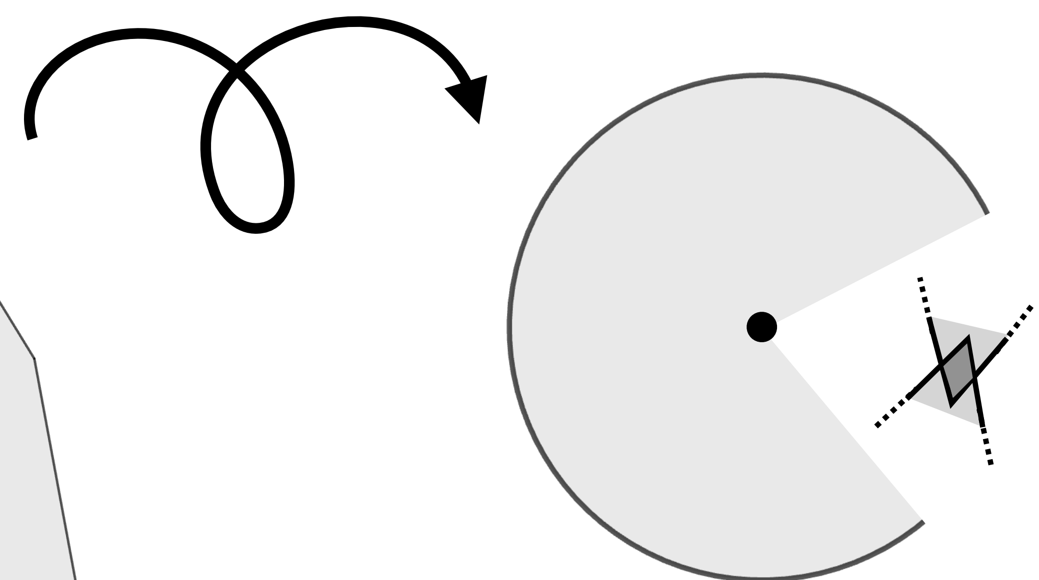

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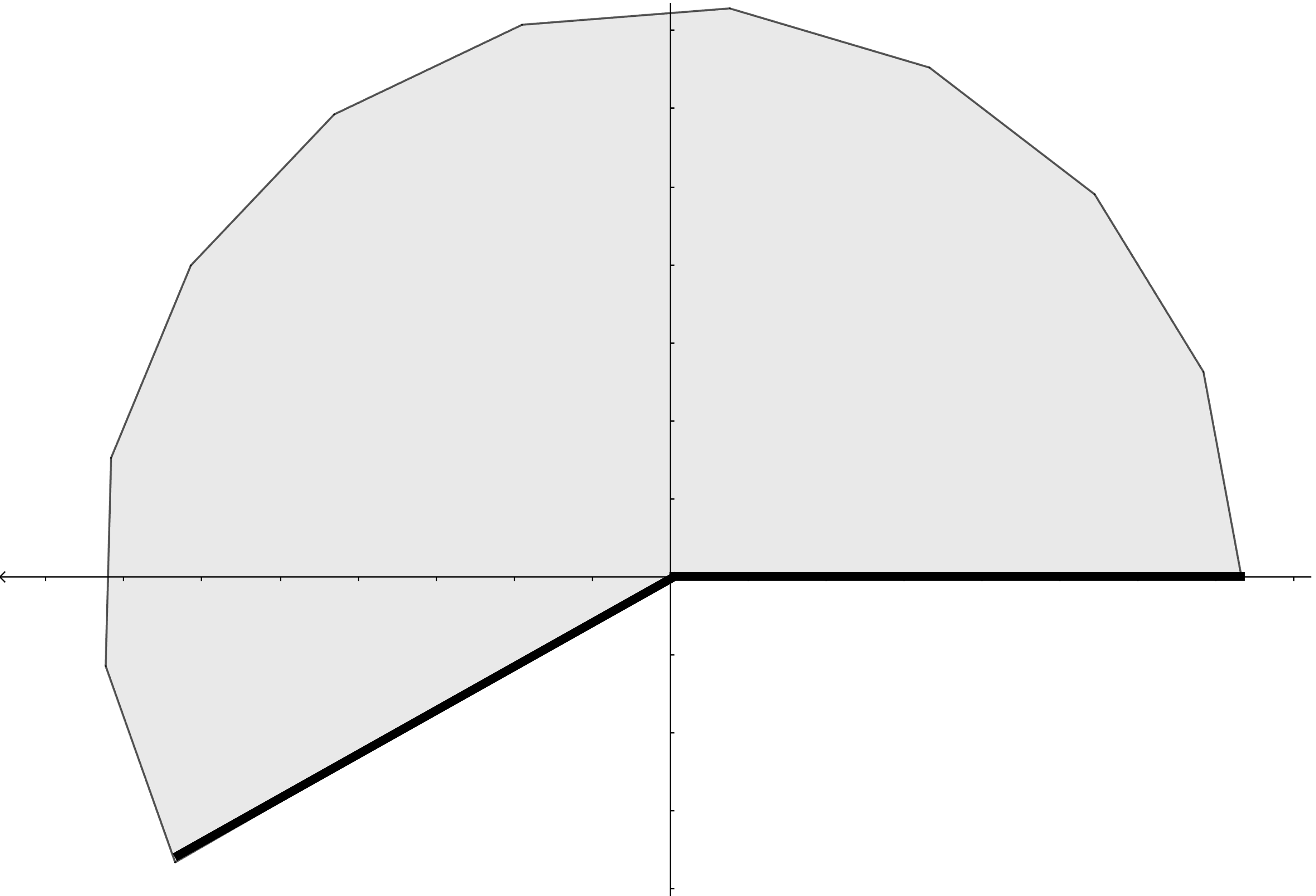

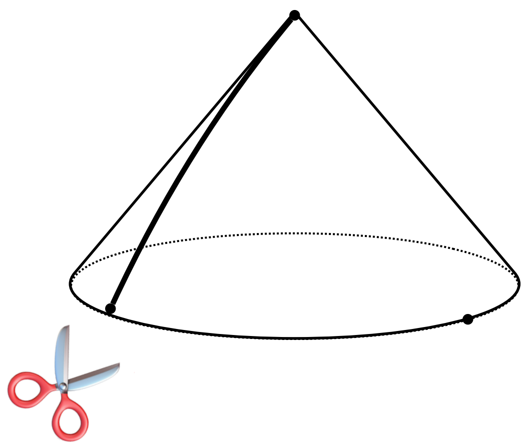

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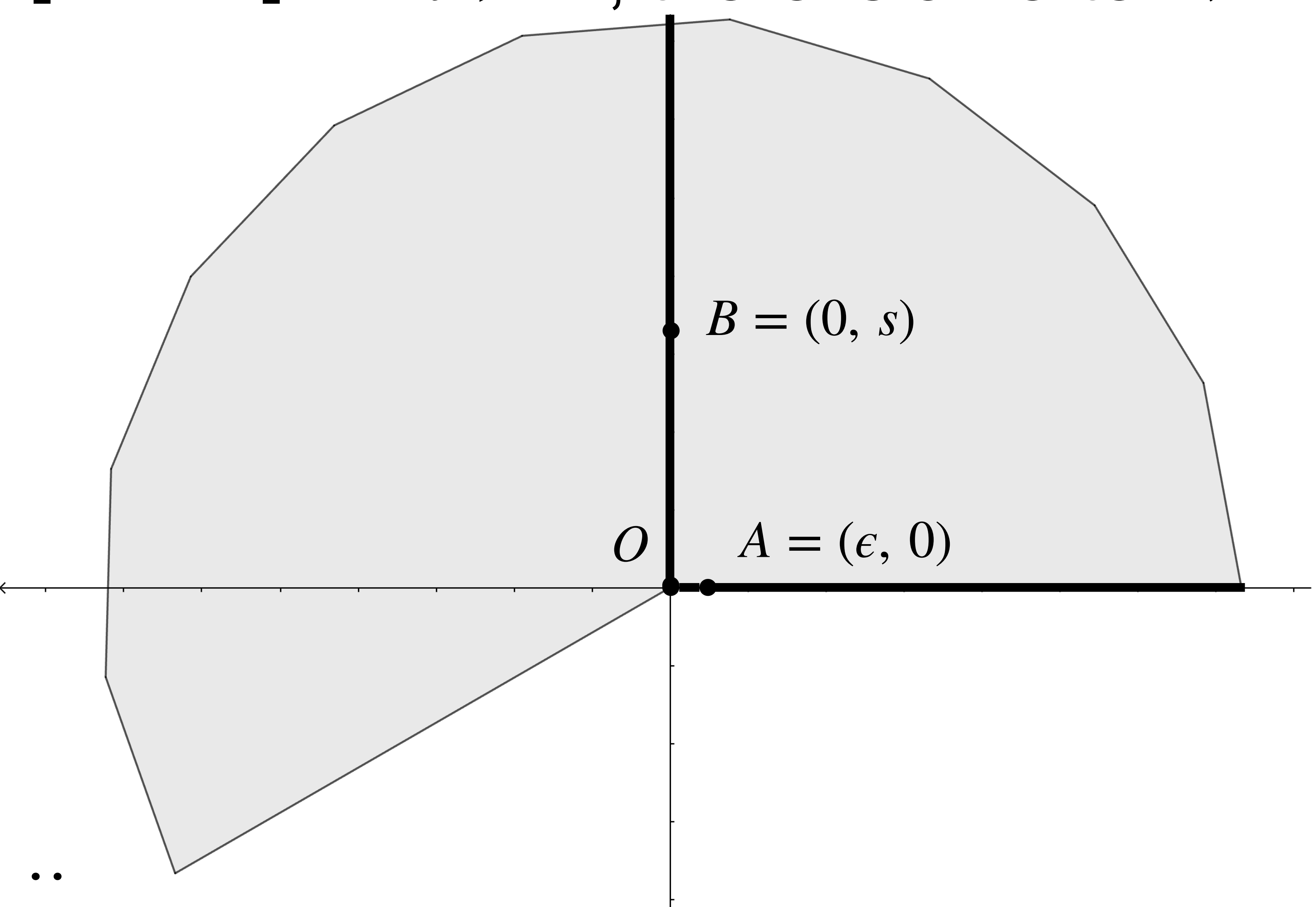

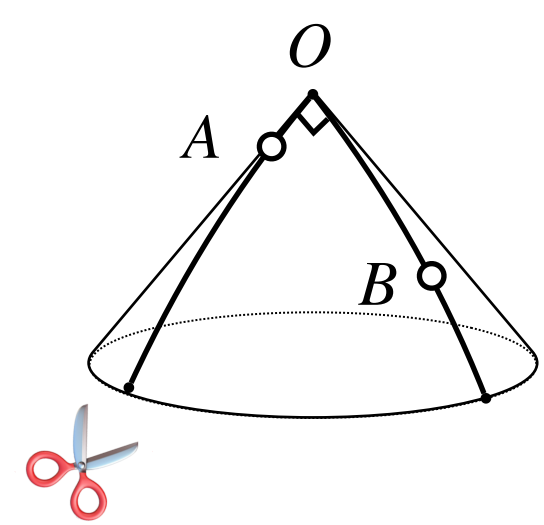

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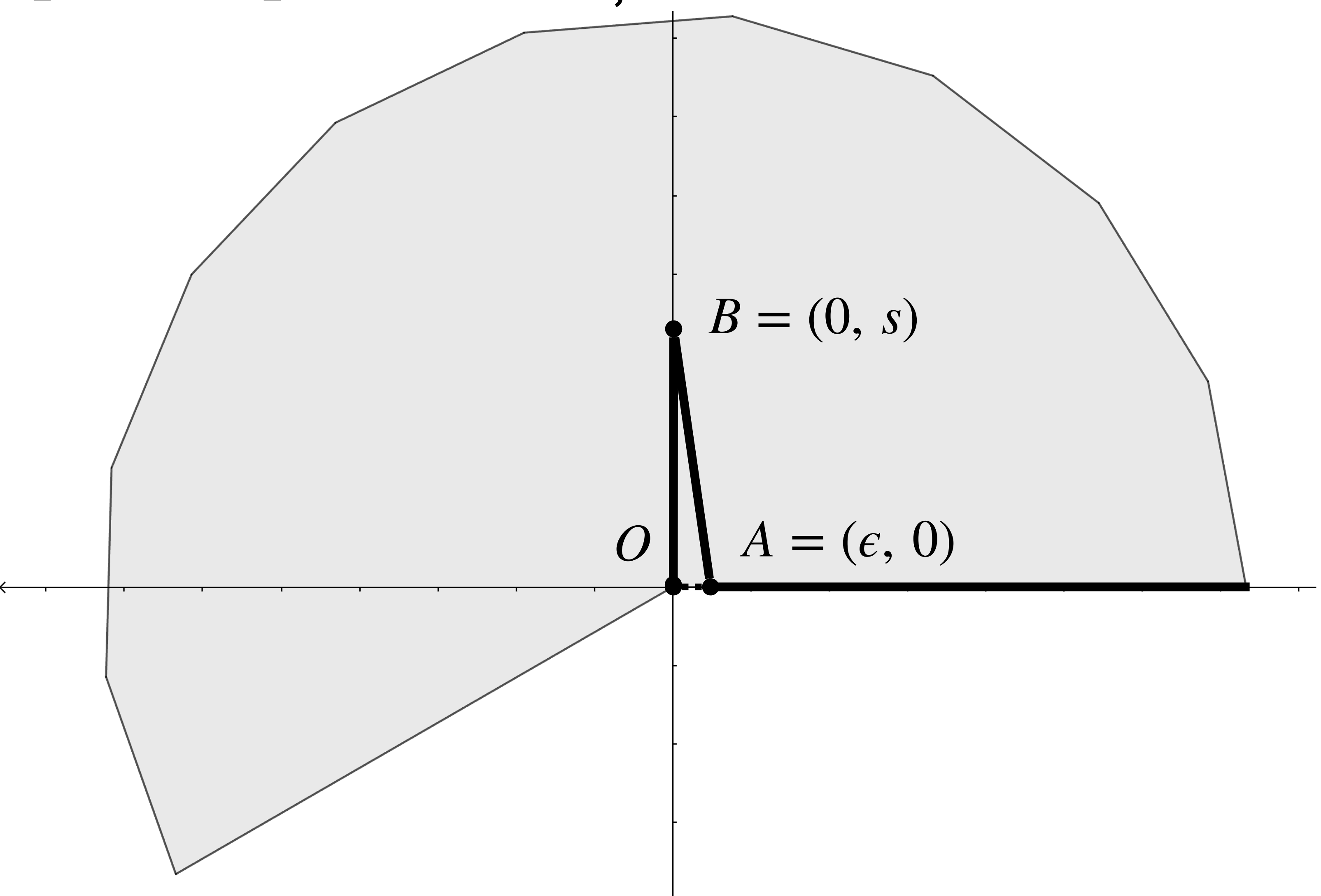

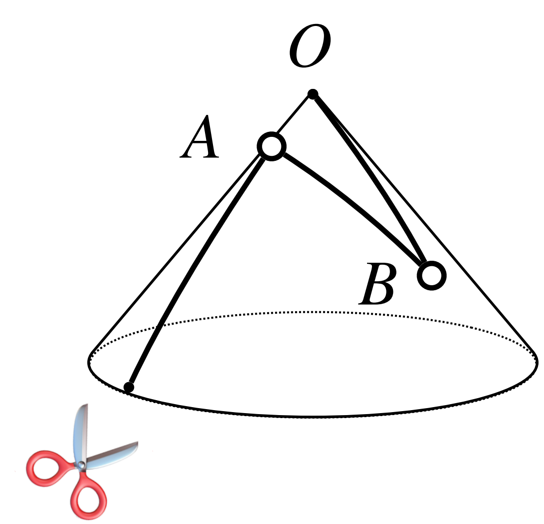

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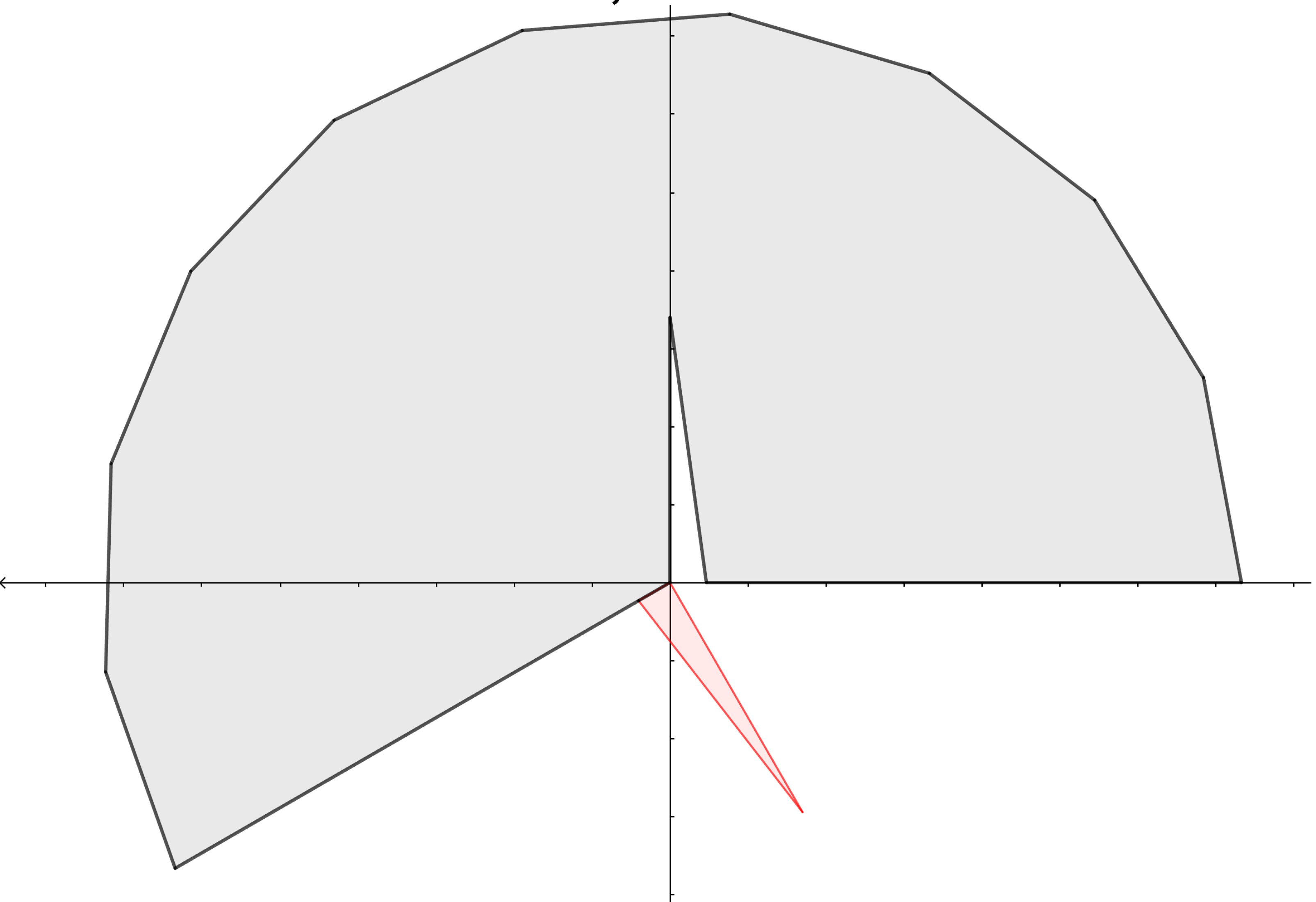

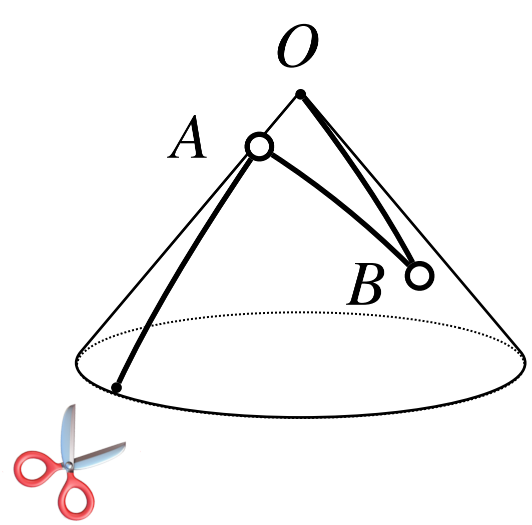

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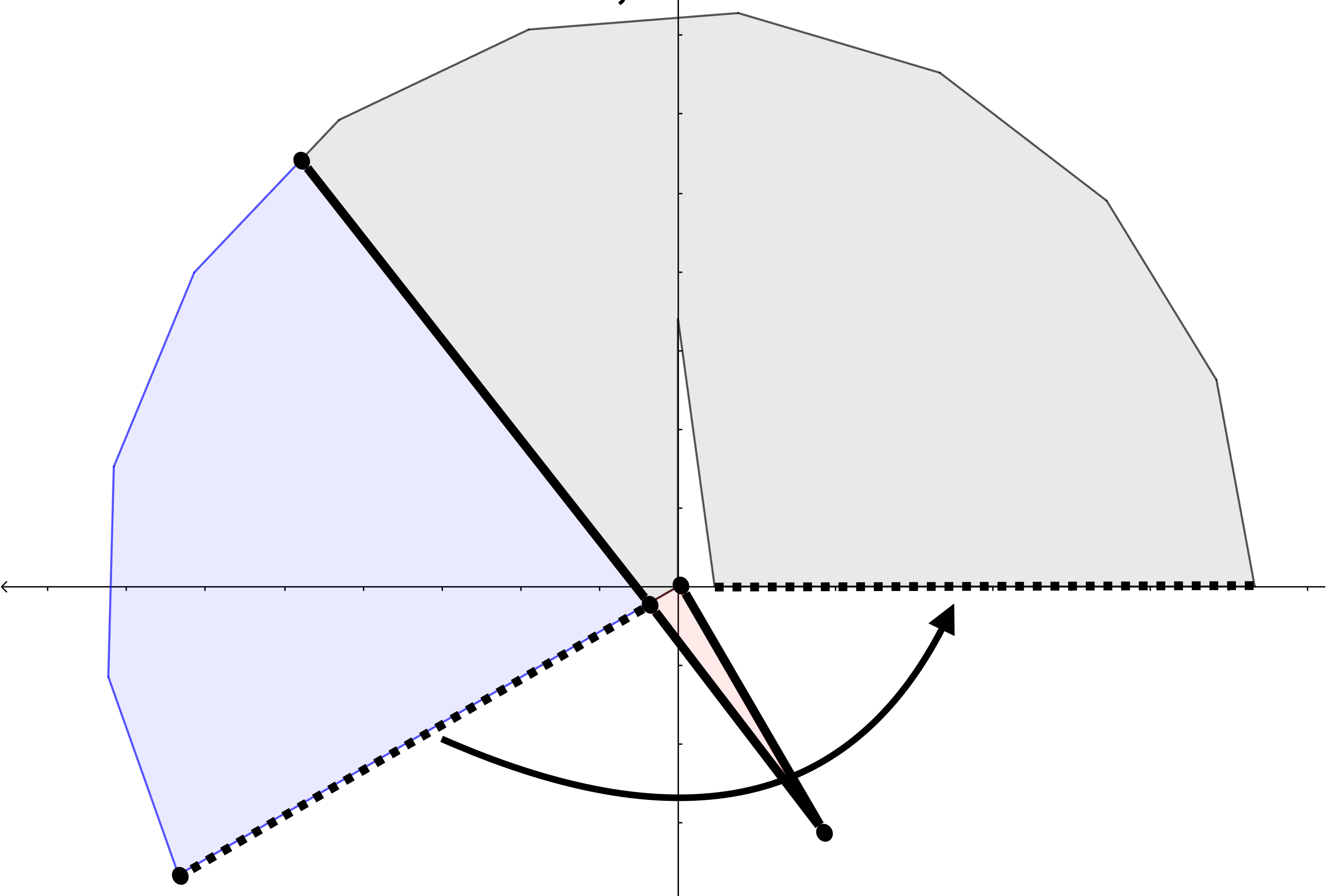

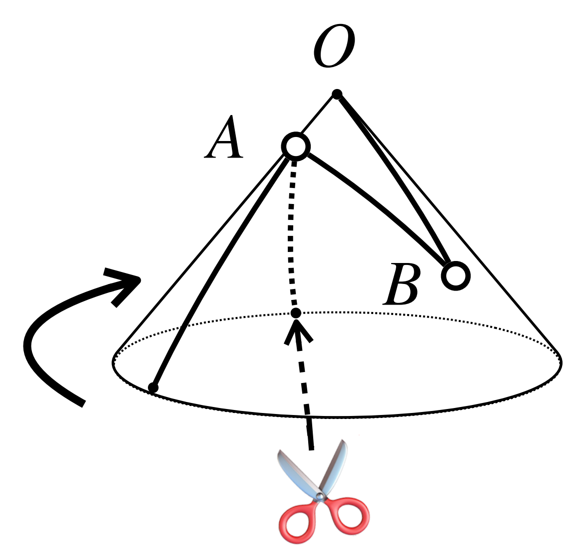

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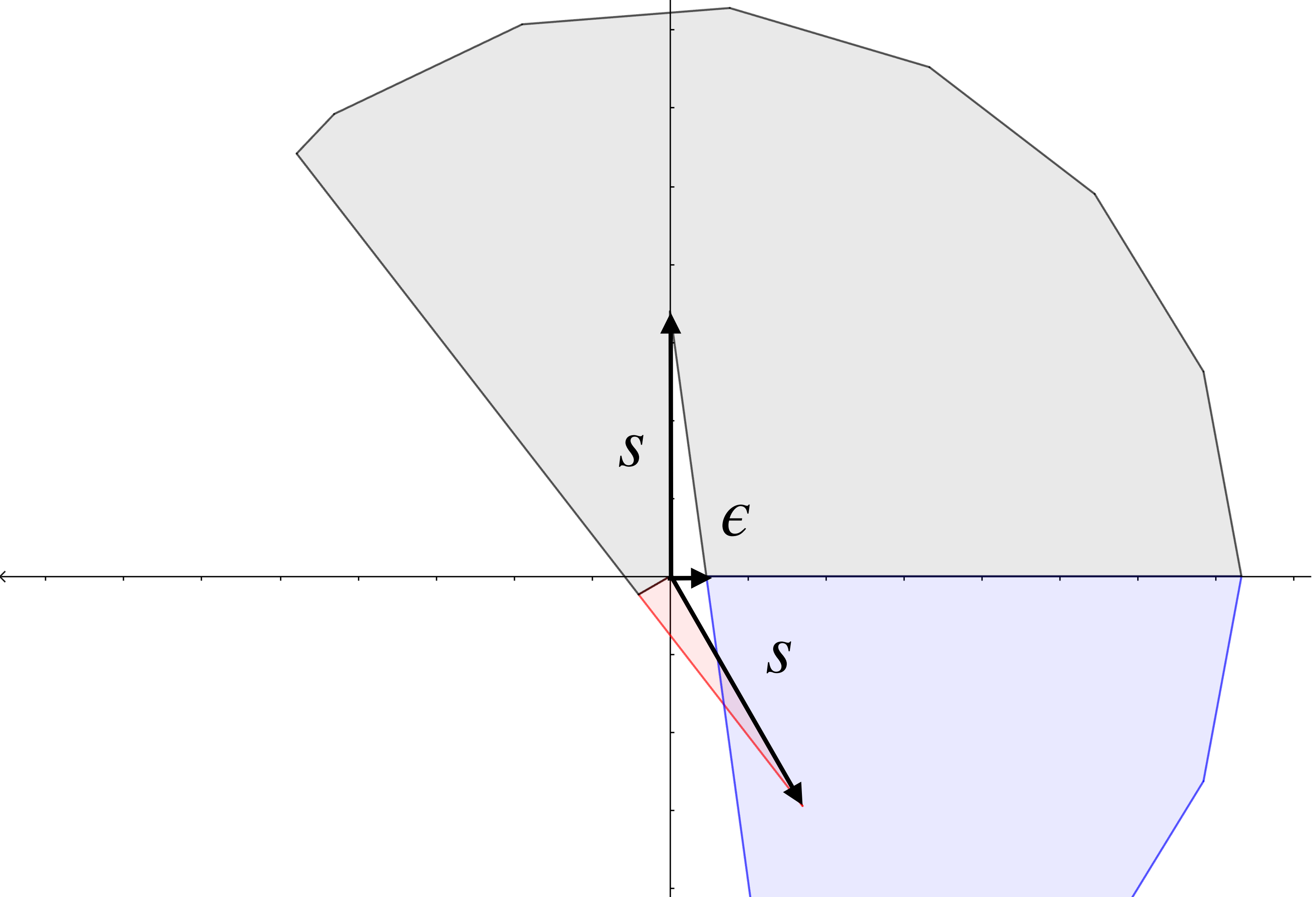

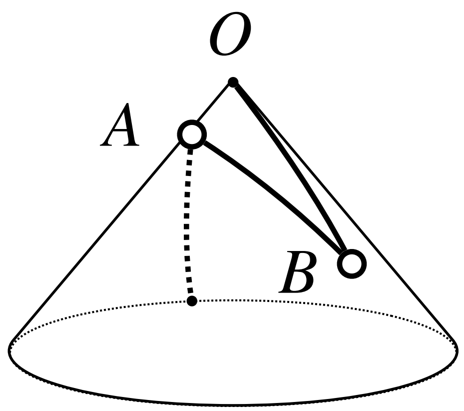

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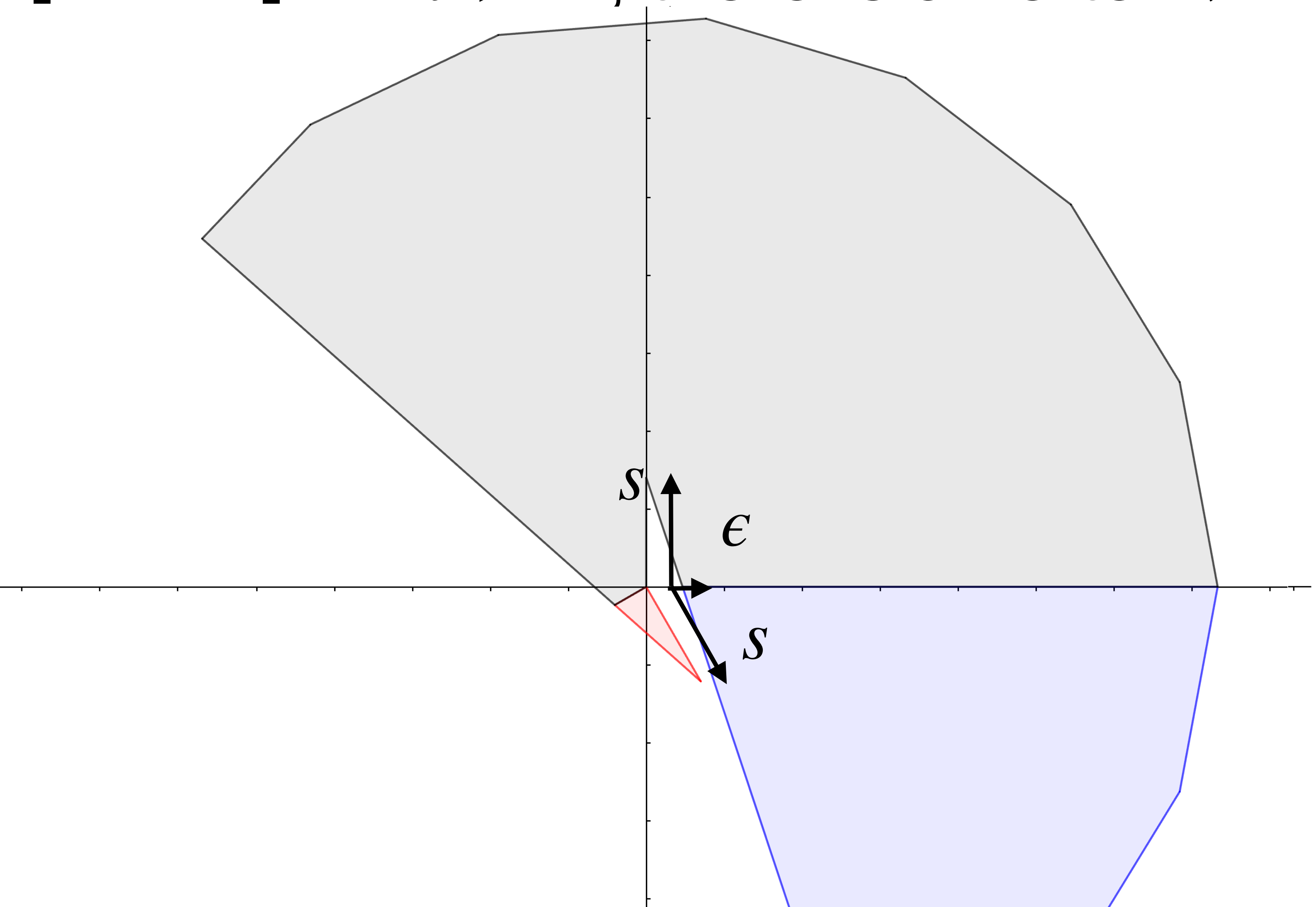

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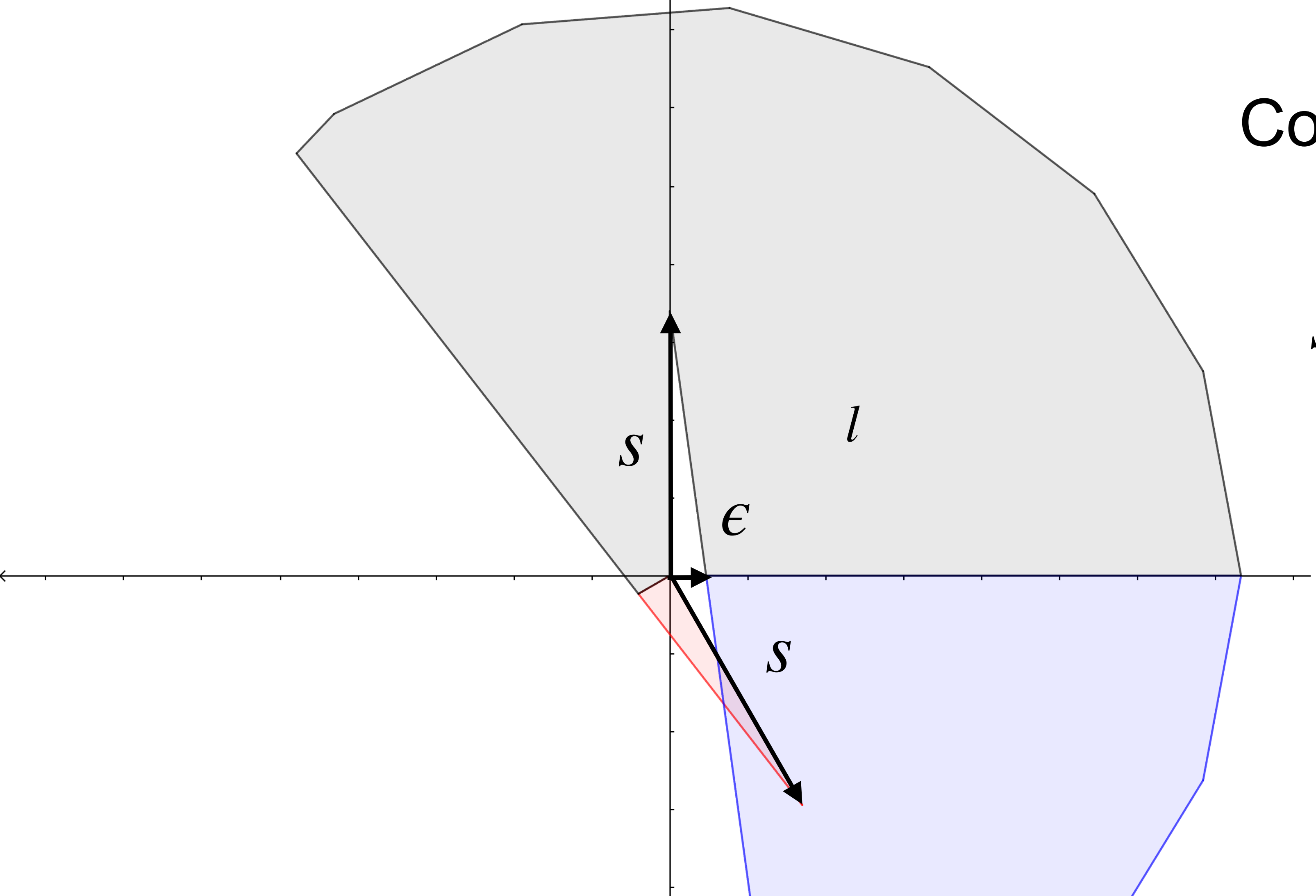
[Proof] If $n > 4$, there is a vertex v which satisfies $\sigma(v) > \pi$



Condition for Overlap:

$$s > \epsilon \cdot \frac{\sin\left(\sigma(v) + \frac{\pi}{2}\right) + 1}{\cos\left(\sigma(v) + \frac{\pi}{2}\right)}$$

Proof of Necessities - Details -

Lemma

If a convex polyhedron Q is not stamper,
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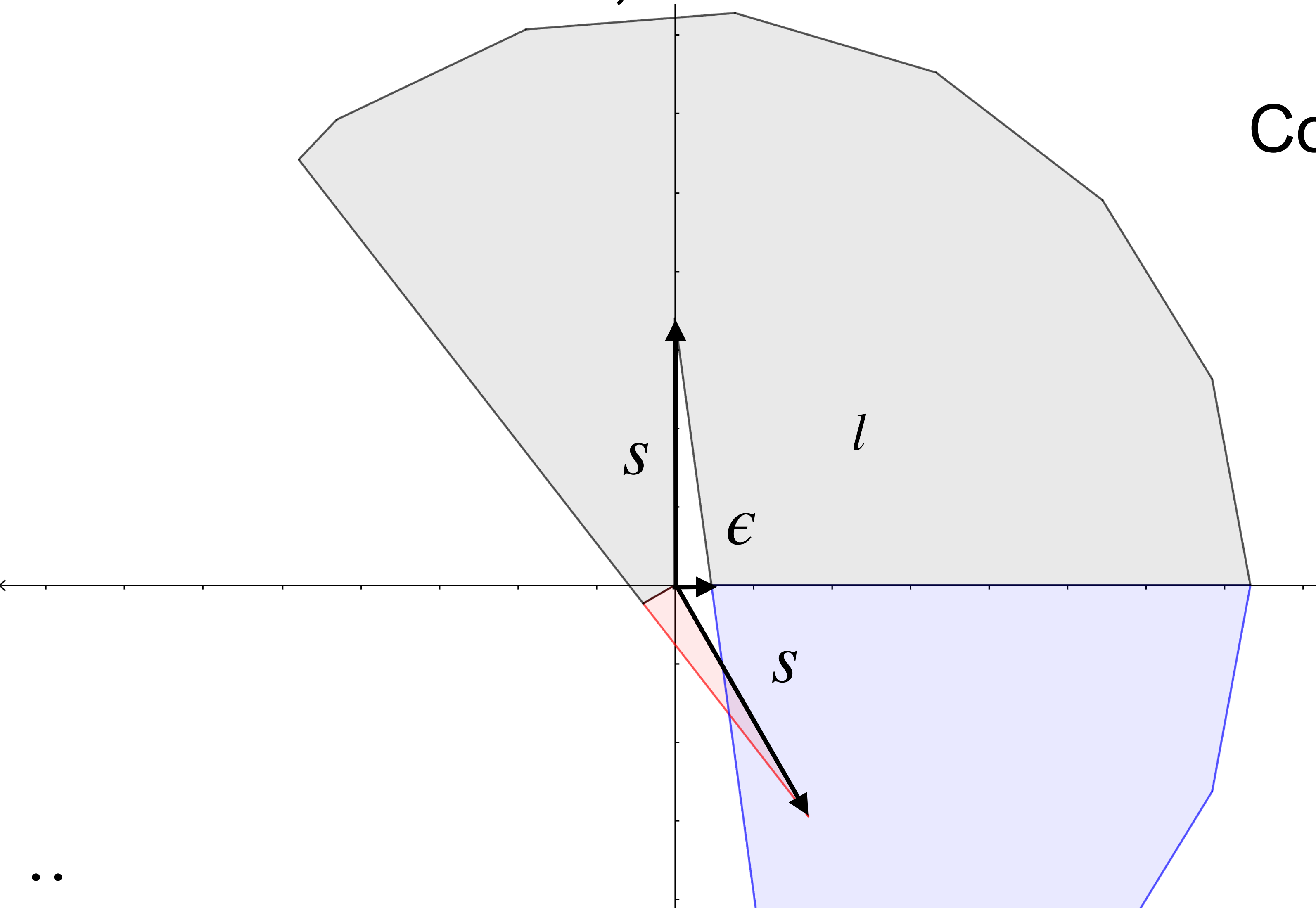

Condition for Overlap:

$$s > \epsilon \cdot \frac{\sin\left(\sigma(v) + \frac{\pi}{2}\right) + 1}{\cos\left(\sigma(v) + \frac{\pi}{2}\right)}$$

By fixing s and making $\epsilon \to 0$,
it can be realized.

□

Proof of Necessities

Lemma

If a convex polyhedron Q is not stamper,
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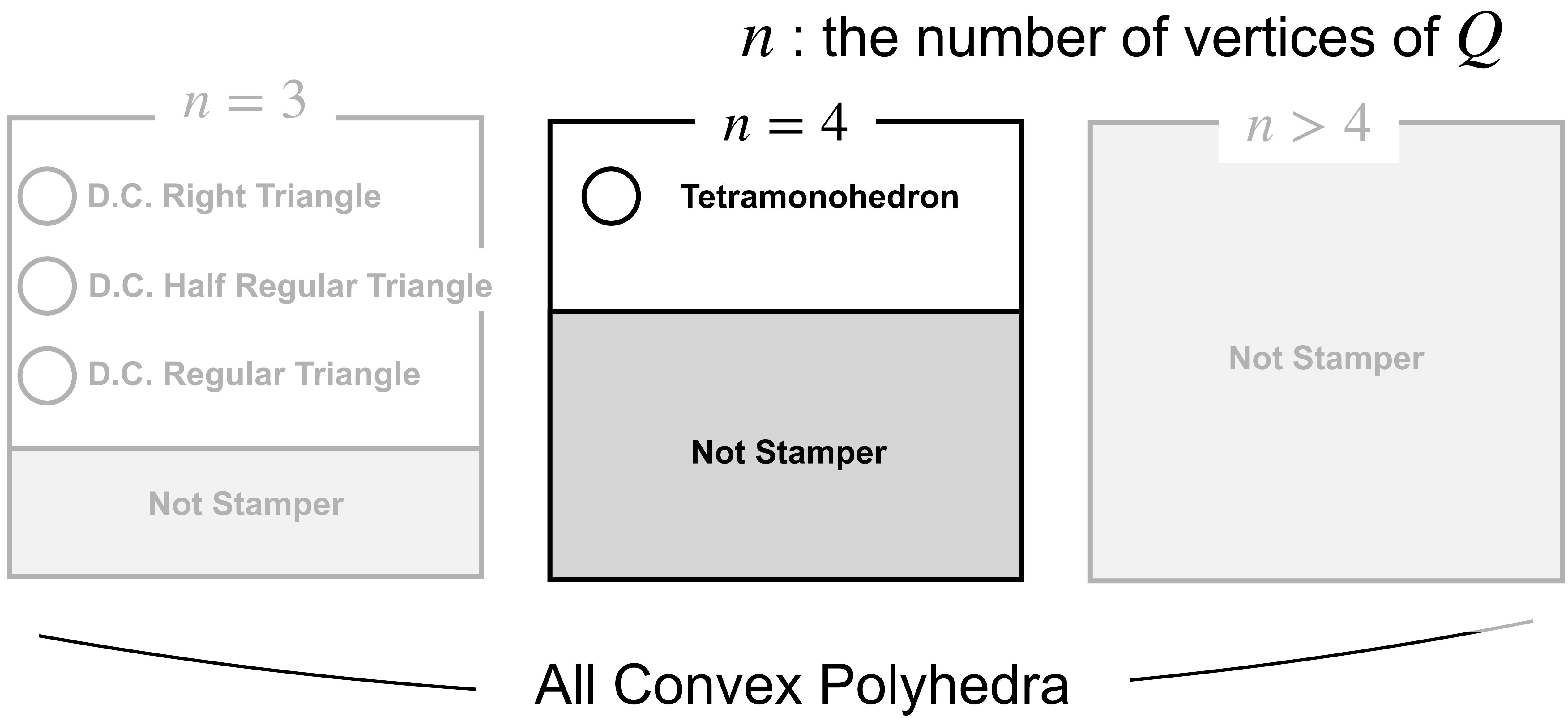

Proof of Necessities - Details -

Lemma

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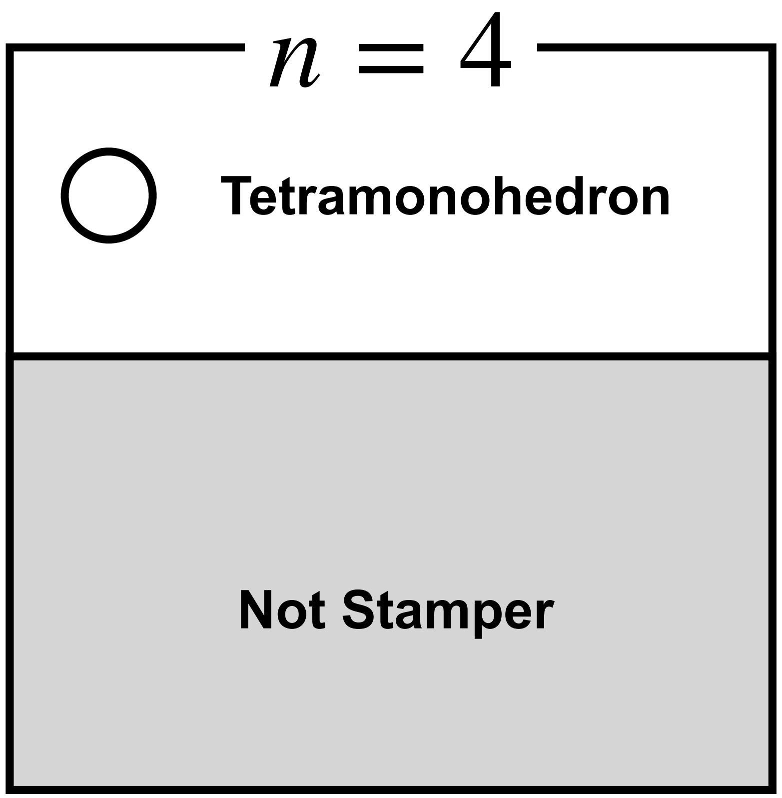
[Proof] Case of $n = 4$



From Descartes' Theorem,
$\sigma(v_1) + \sigma(v_2) + \sigma(v_3) + \sigma(v_4) = 4\pi$

The average of $\sigma(v_i)$ is π

For at least one v_i,
$\pi < \sigma(v_i)$

For any v_i,
$\sigma(v_i) = \pi$

Proof of Necessities - Details -

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○ Tetramonohedron

Not Stamper

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Proof of Necessities - Details -

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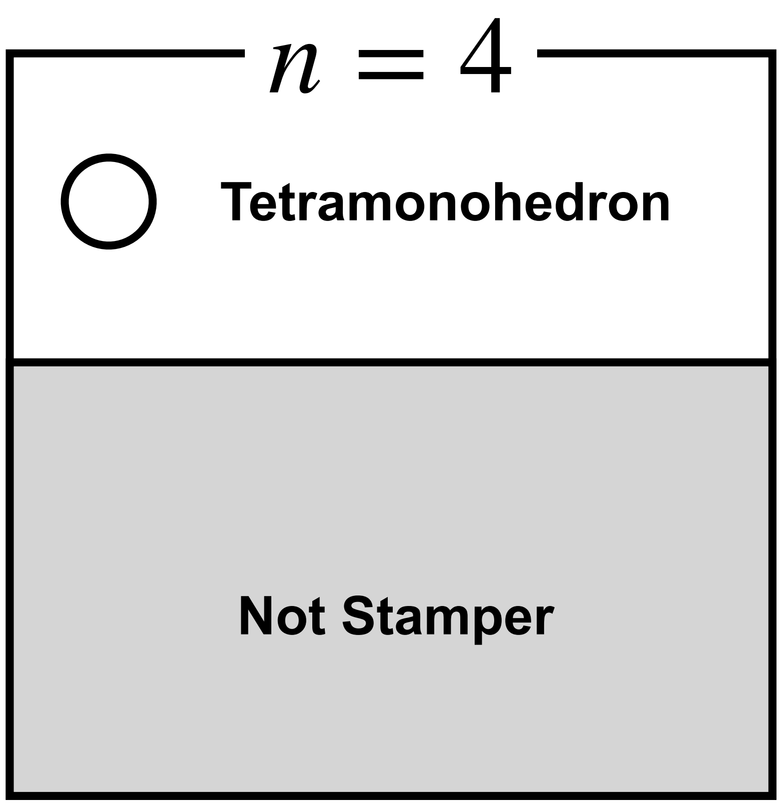

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Q is a tetramonohedron

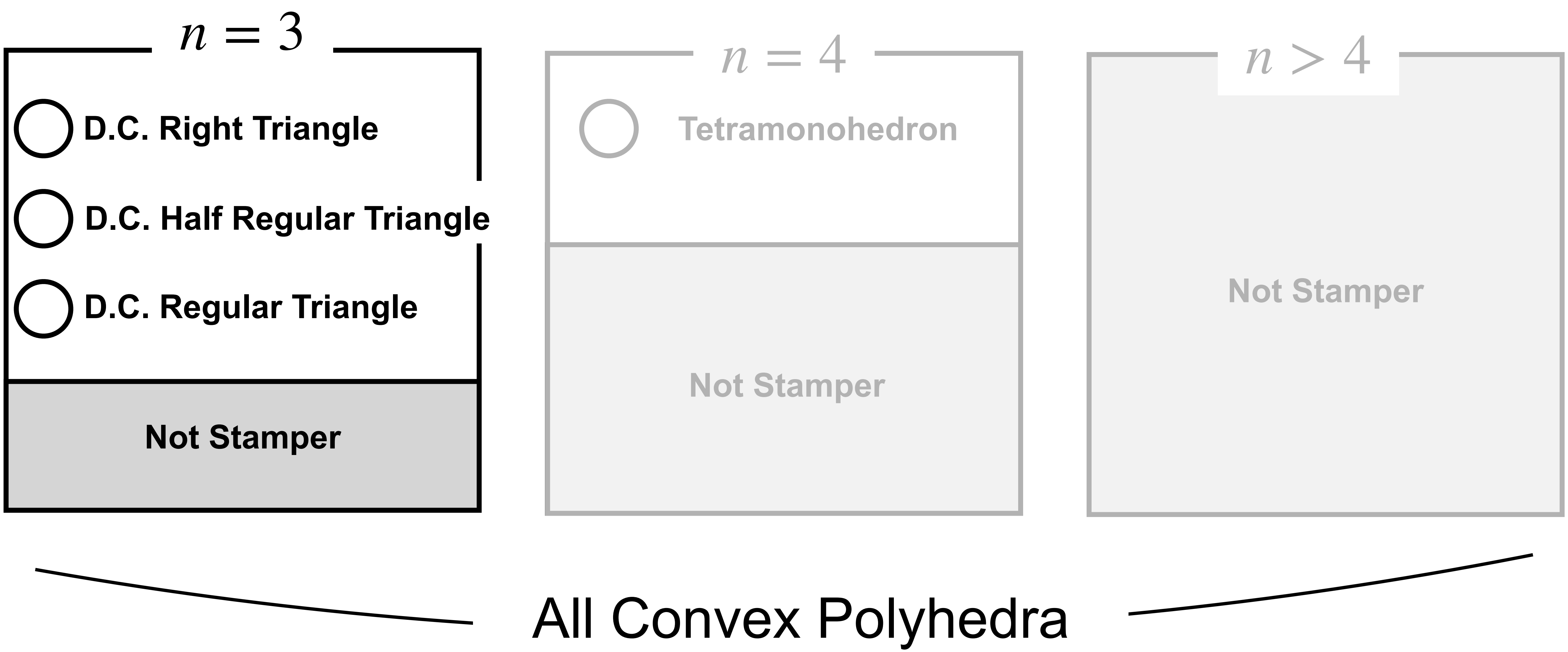
Reduced to case of $n > 4$ □

Proof of Necessities

Lemma

If a convex polyhedron Q is not stamper,
Q has overlapping unfolding.

n : the number of vertices of Q



Proof of Necessities

- Details -

Lemma

If a convex polyhedron Q is not stamper,
Q has overlapping unfolding.

[Proof] Case of $n = 3$

$n = 3$

- D.C. Right Triangle
- D.C. Half Regular Triangle
- D.C. Regular Triangle

Not Stamper

From Descartes' Theorem,
$\sigma(v_1) + \sigma(v_2) + \sigma(v_3) = 2\pi$ → Stamper

For at least one v_i: $\frac{\pi}{2} < \sigma(v) < \frac{2\pi}{3}$

For at least one v_i: $\frac{2\pi}{3} < \sigma(v) < \pi$

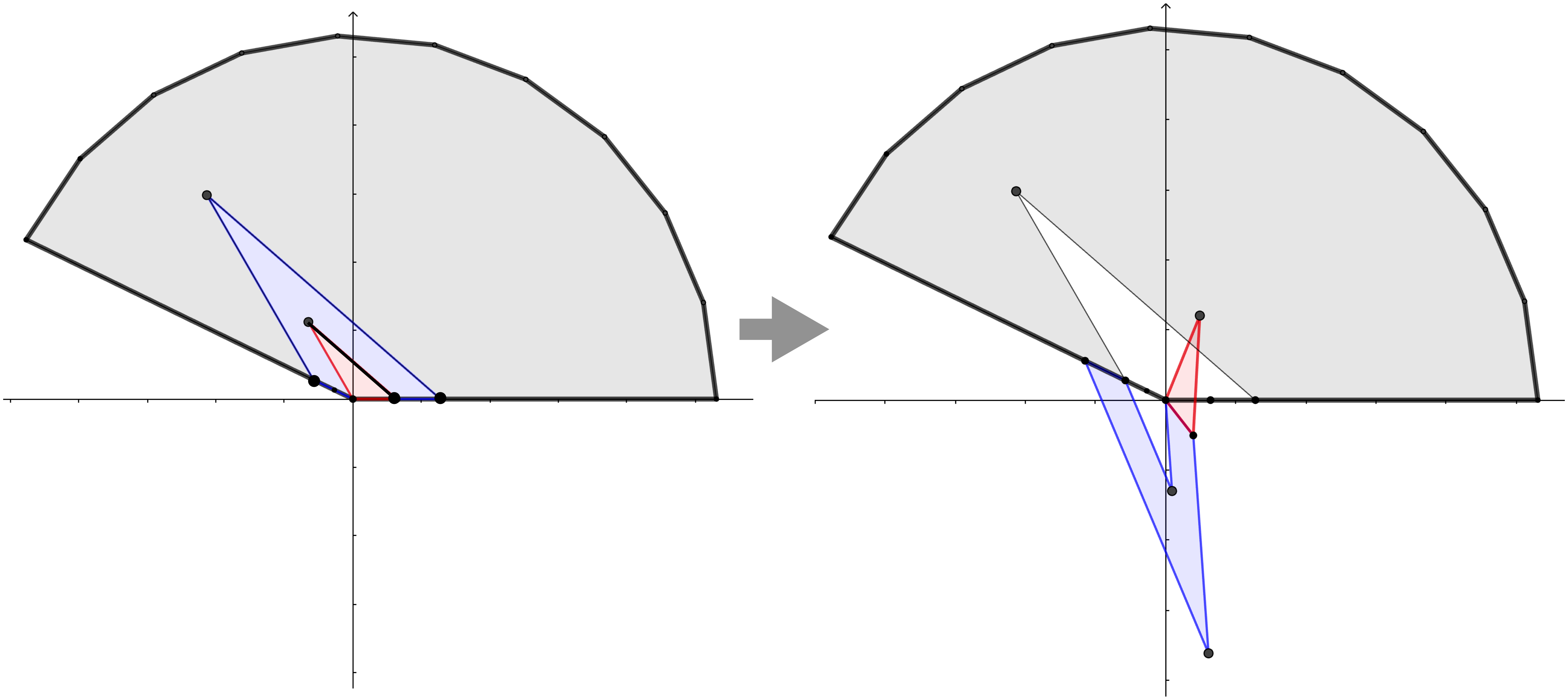
For at least one v_i: $\pi < \sigma(v_i)$

Reduced to case of $n > 4$

..

Proof of Necessities

Case that there is a vertex v_i which satisfies $\frac{2\pi}{3} < \sigma(v) < \pi$



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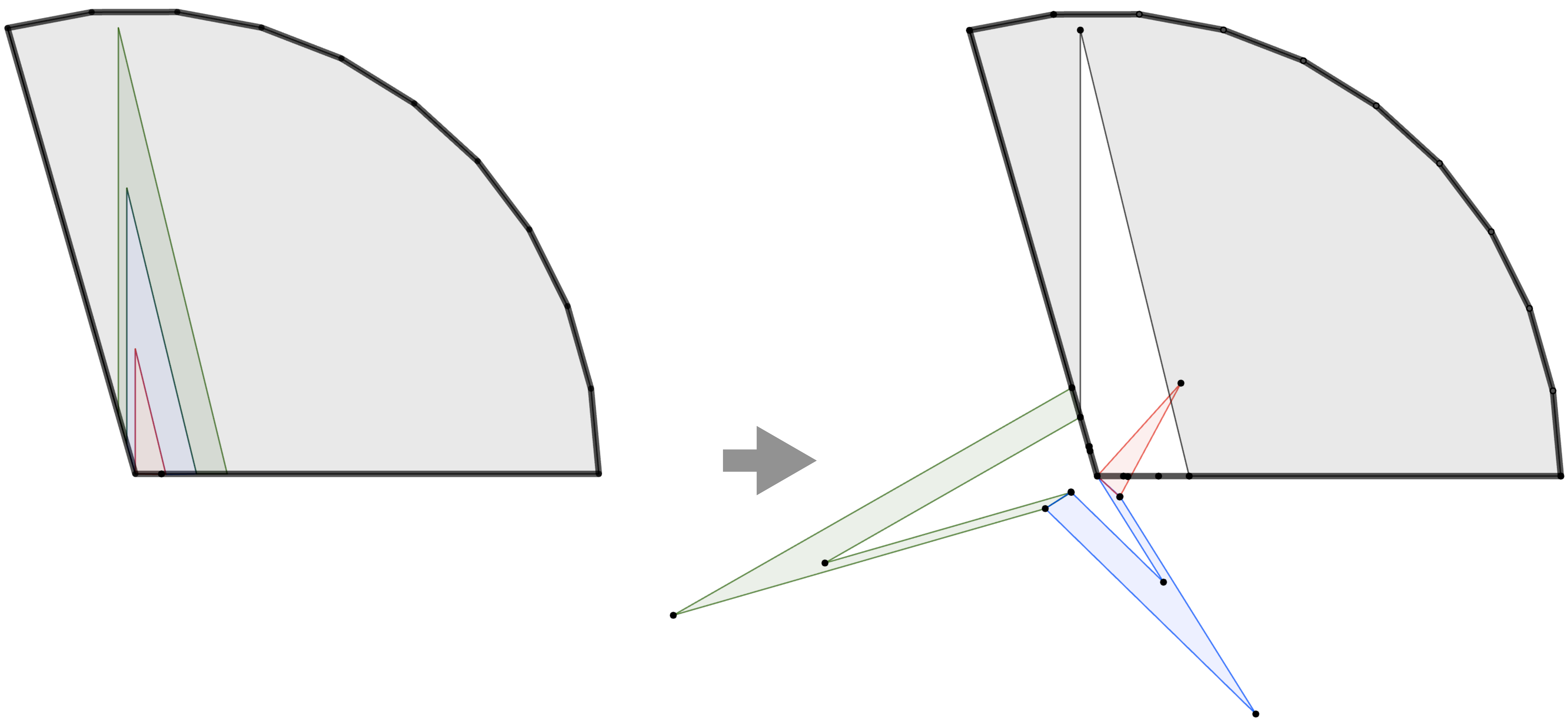

□

Summary and Future Work

Theorem

For any convex polyhedron Q,

Q is overlap-free

$\Updownarrow$

Q is either one of

- tetramonohedron
- doubly-covered regular triangle
- doubly-covered half regular triangle
- doubly-covered right triangle

…

Summary and Future Work

Theorem

For any convex polyhedron Q,

Q is overlap-free $\Leftrightarrow$ Q is “stamper”

[Future Work]

By extending the “Overlap-free”,

we can consider a concept of

“Any edge unfolding has no overlaps”

(= Edge-overlap-free)

…

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➡ What kinds of polyhedra are edge-overlap-free?

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Fin.