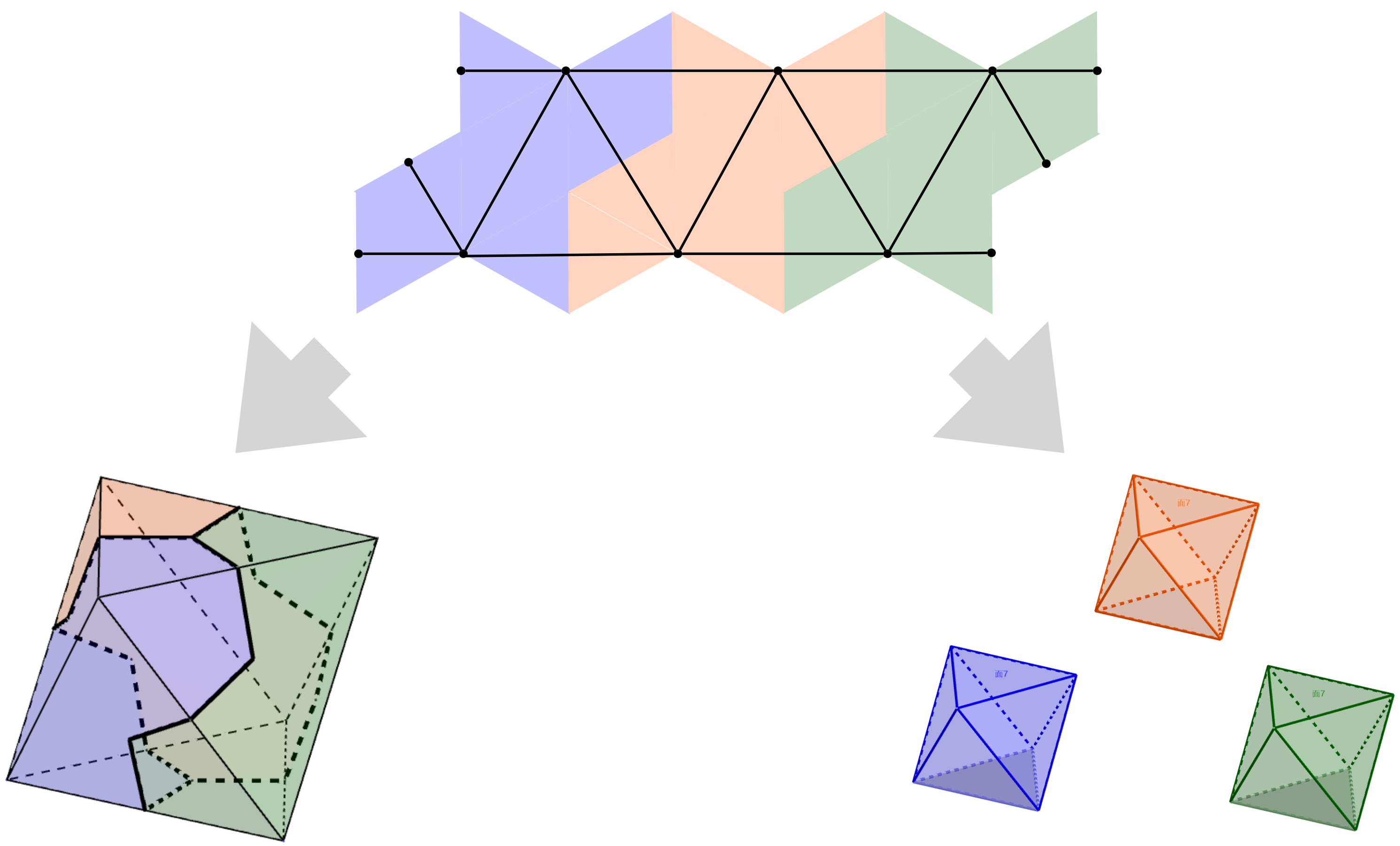
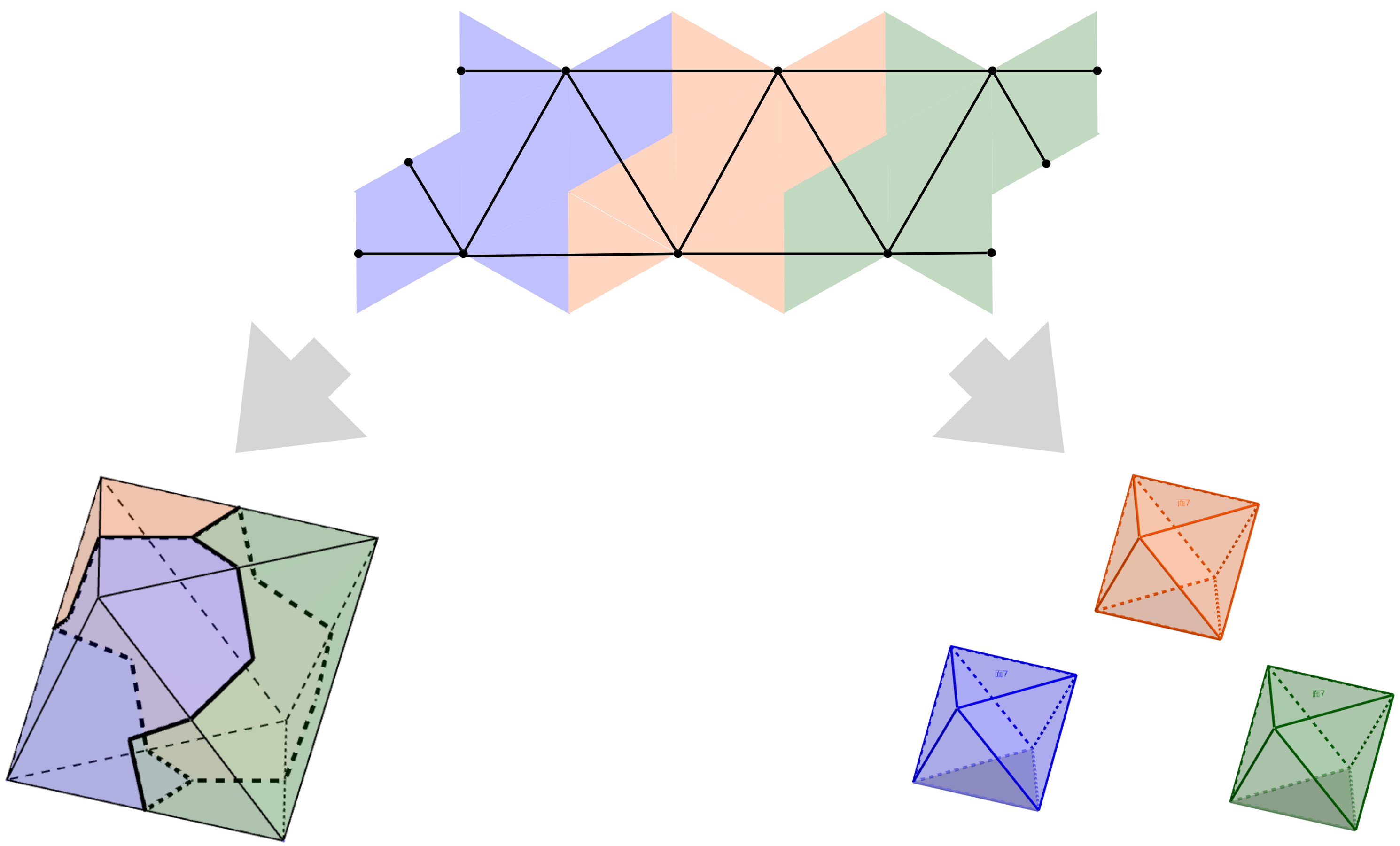
Dissections of a Net of a Regular Octahedron into Nets of Regular Octahedra



Yuta Nomi (JAIST)
Takumi Shiota (Kyutech)
Tonan Kamata (JAIST)
Ryuhei Uehara (JAIST)

The 26th Japan Conference on Discrete and Computational Geometry, Graphs, and Games
10th - 12th, September, 2024

Dissections of a Net of a Regular Octahedron into Nets of Regular Octahedra

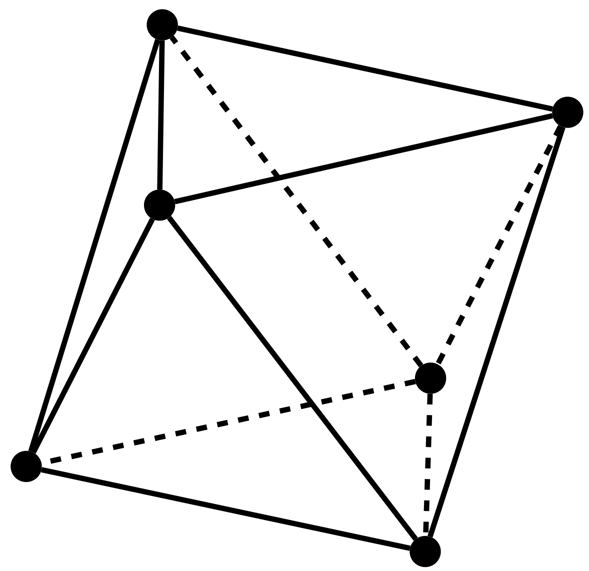


Yuta Nomi (JAIST)
Takumi Shiota (Kyutech)
Tonan Kamata (JAIST)
Ryuhei Uehara (JAIST)

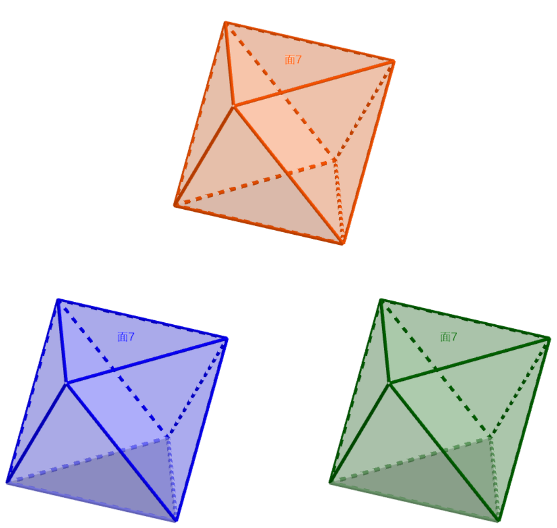
The 26th Japan Conference on Discrete and Computational Geometry, Graphs, and Games
10th - 12th, September, 2024

Our Target

Single Regular Octahedron



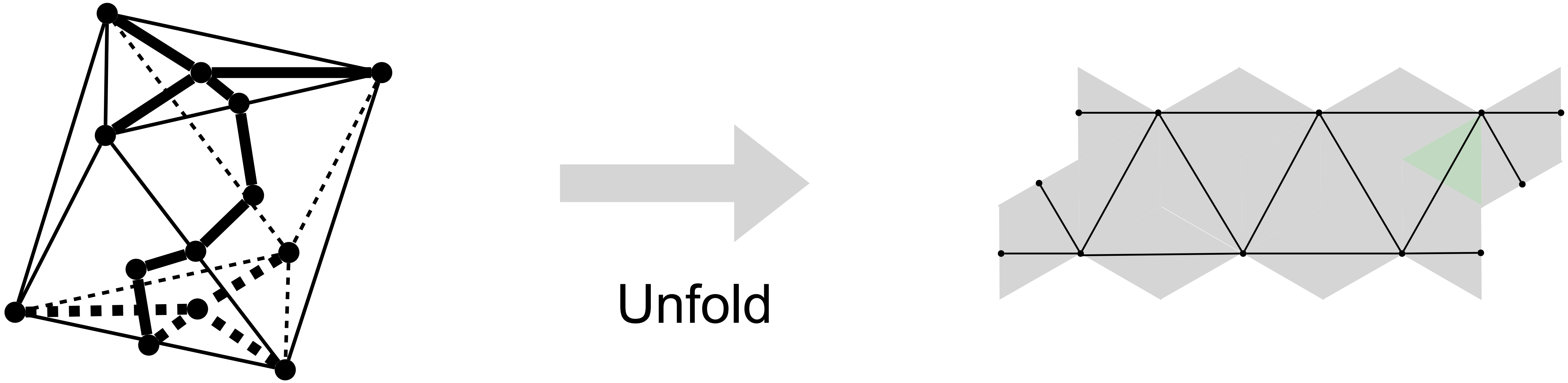
Replicate



Multiple Regular Octahedra

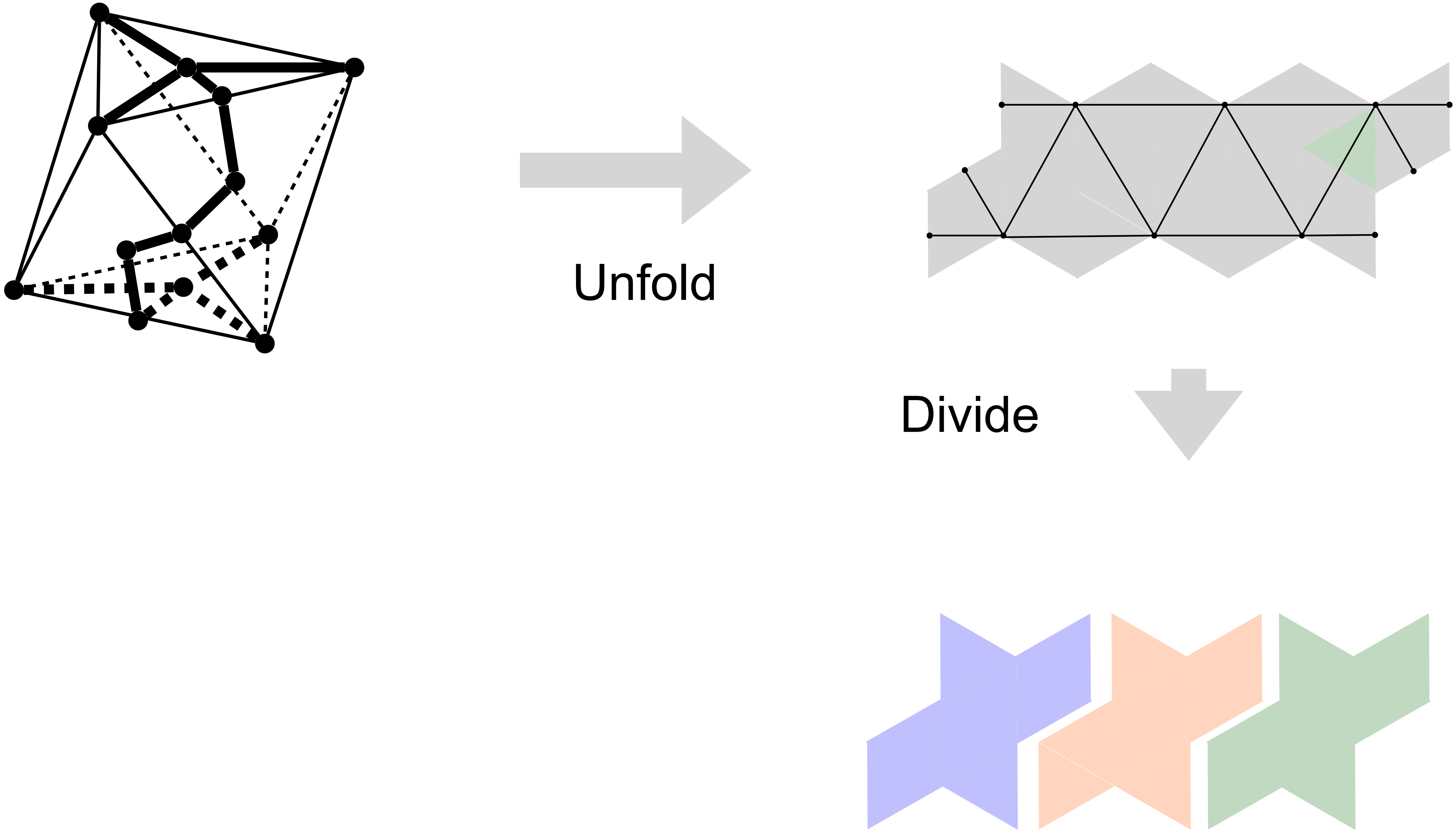
Our Target

Single Regular Octahedron



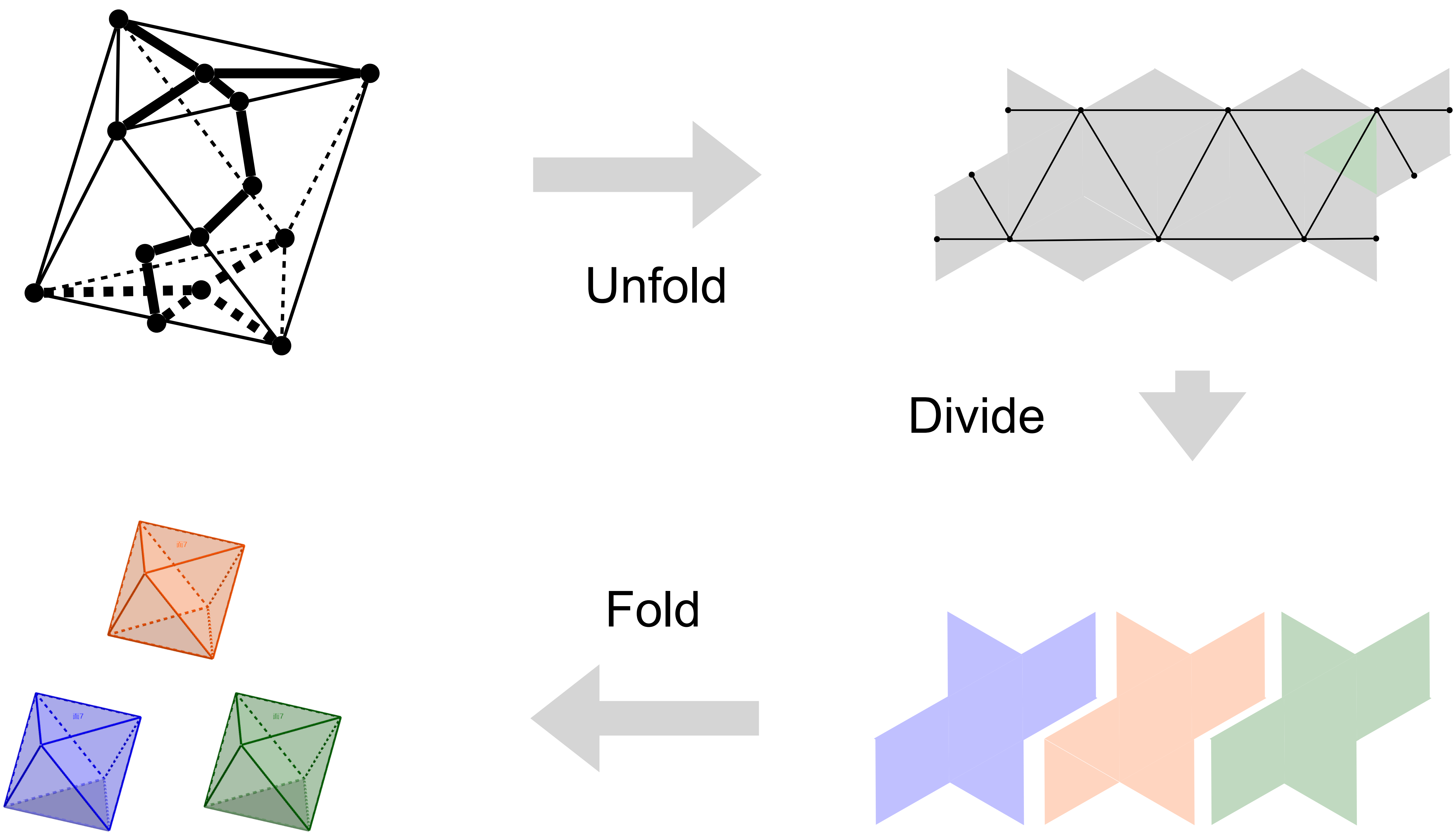
Our Target

Single Regular Octahedron



Our Target

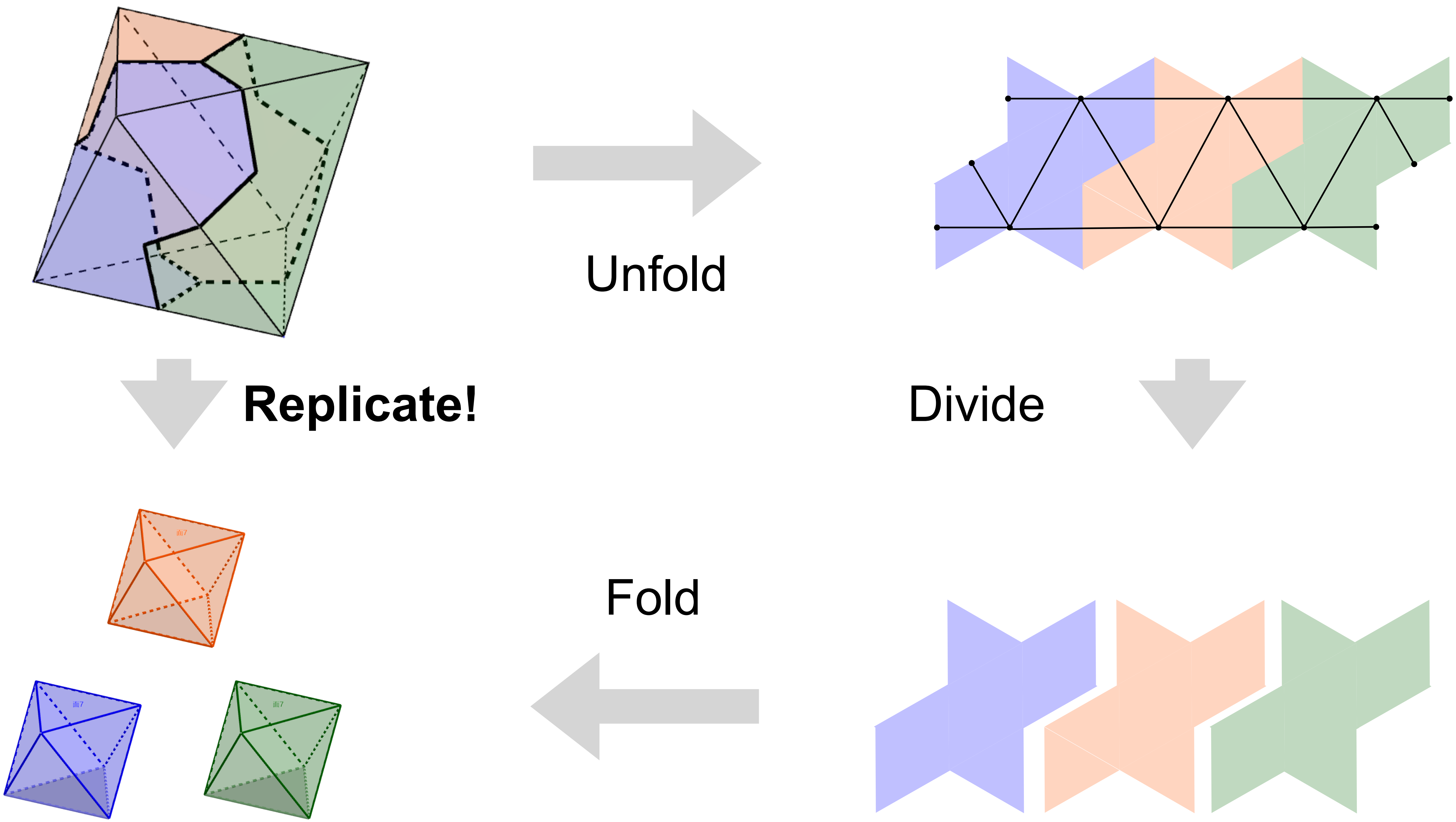
Single Regular Octahedron



Multiple Regular Octahedra

Our Target

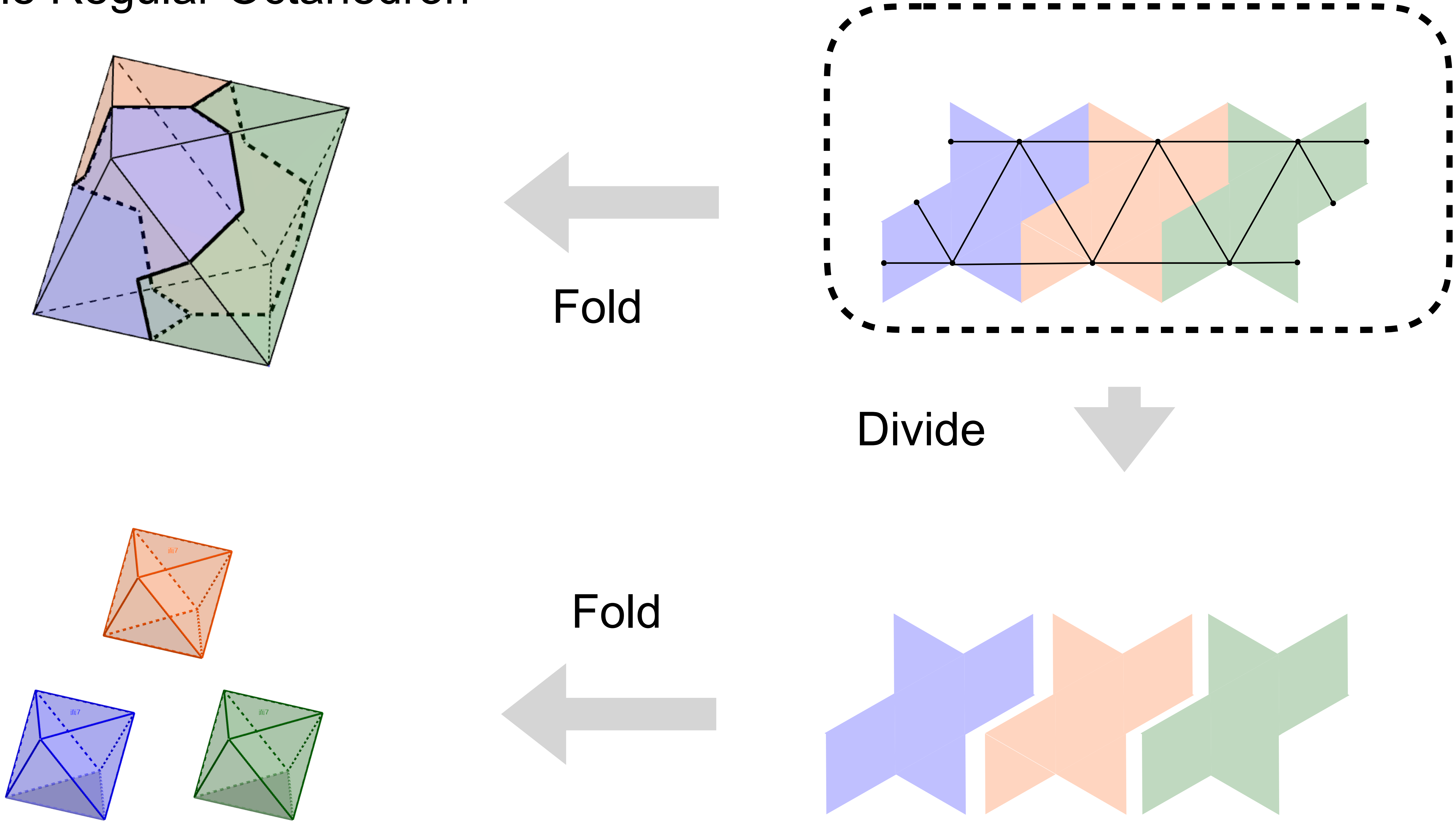
Single Regular Octahedron



Multiple Regular Octahedron

Our Target

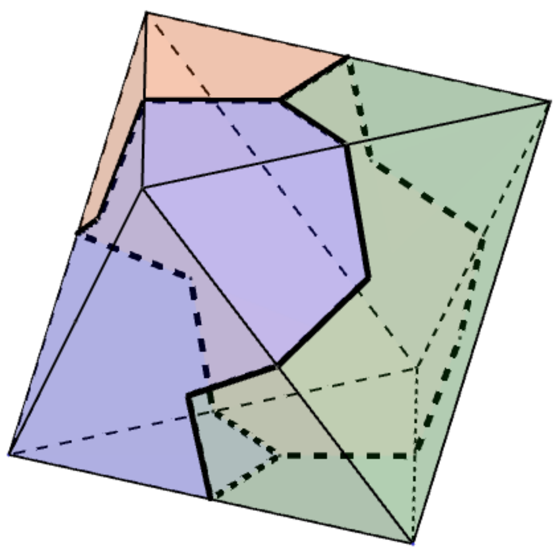
Single Regular Octahedron



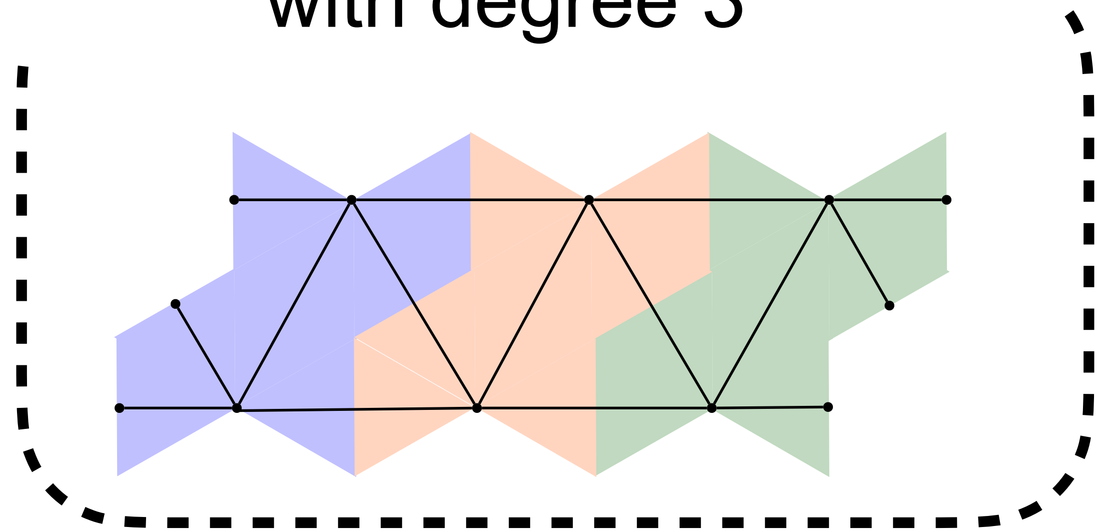
Multiple Regular Octahedra

Our Target

Single Regular Octahedron



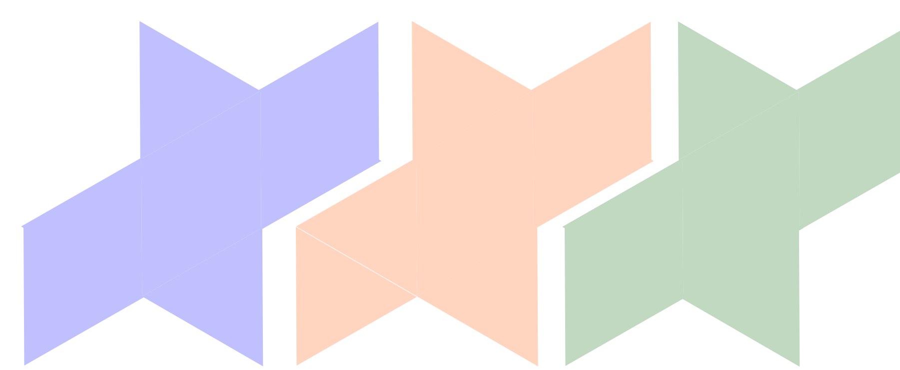
Replicative Net of Octahedron
(= **Rep-oct.**)
with degree 3

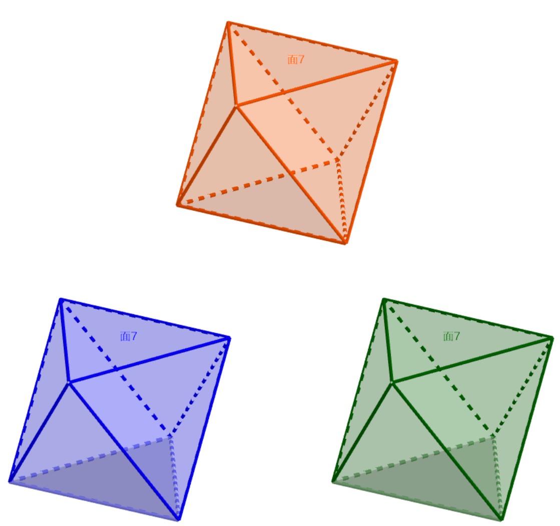


Fold

Divide

Fold

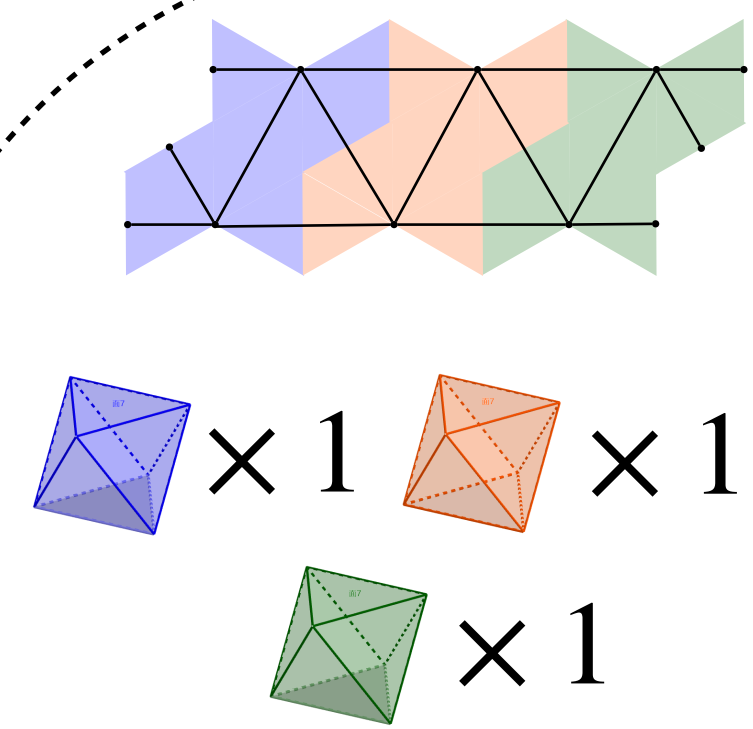




Multiple Regular Octahedra

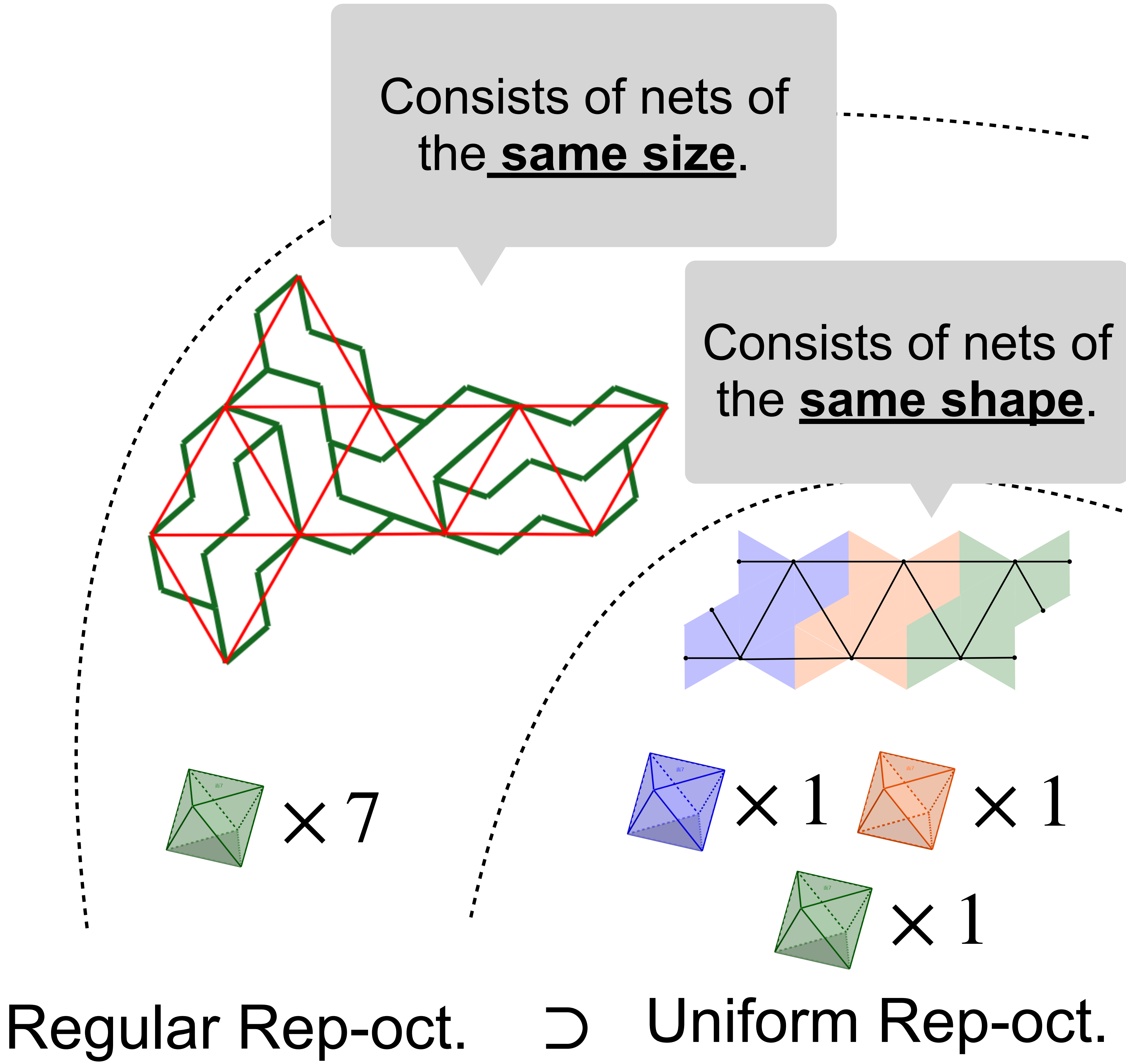
Notation

Consists of nets of the **same shape**.



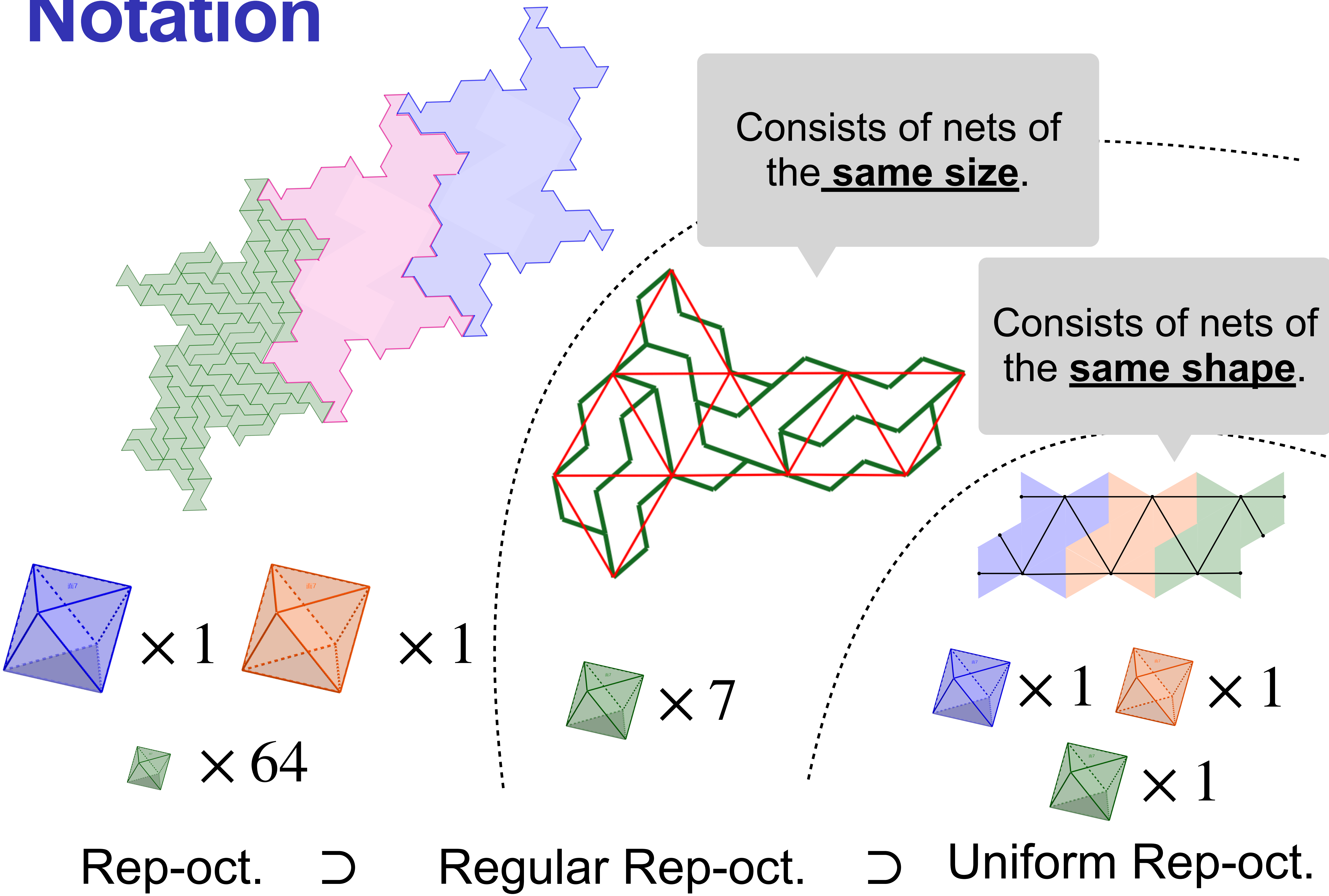
Uniform Rep-oct.

Notation



Consists of nets of the **same size**.
Consists of nets of the **same shape**.
× 7
× 1
× 1
× 1
Regular Rep-oct. ⊃ Uniform Rep-oct.

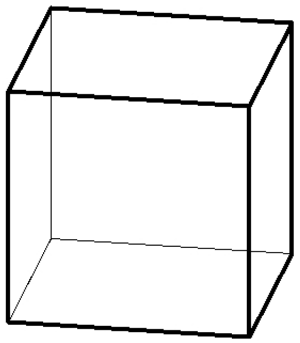
Notation



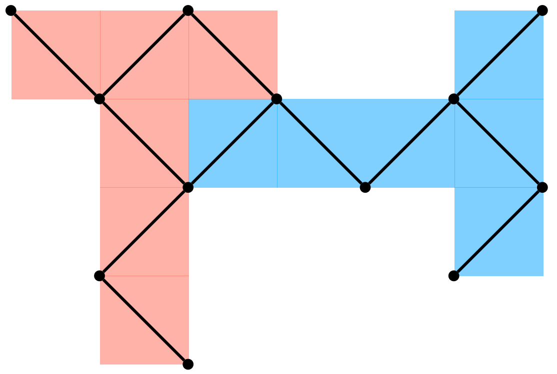
Consists of nets of the **same size**.
Consists of nets of the **same shape**.
× 1
× 1
× 64
× 7
× 1
× 1
× 1
Rep-oct. ⊃ Regular Rep-oct. ⊃ Uniform Rep-oct.

Background

[1] Zachary Abel, Brad Ballinger, Erik D. Demaine, Martin L. Demaine, Jeff Erickson, Adam Hesterberg, Hiro Ito, Irina, Kostitsyna, Jayson Lynch, and Ryuhei Uehara. Unfolding and dissection of multiple cubes, tetrahedra, and doubly covered squares. J. Inf. Process., 25:610–615, 2017.



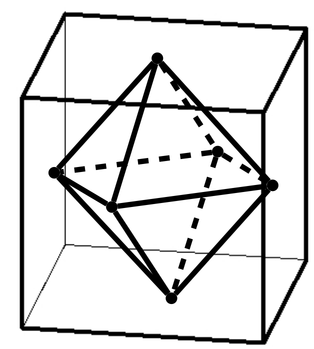
Cube



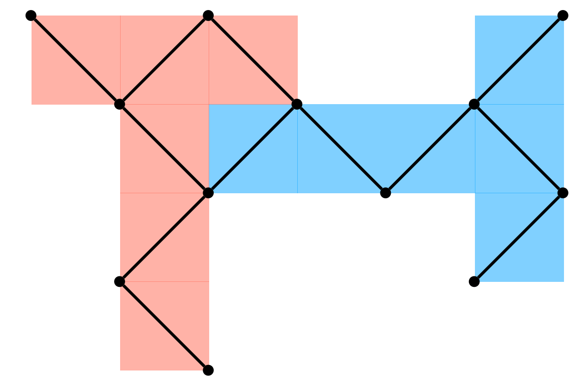
Rep-cube [1]

Background

[1] Zachary Abel, Brad Ballinger, Erik D. Demaine, Martin L. Demaine, Jeff Erickson, Adam Hesterberg, Hiro Ito, Irina, Kostitsyna, Jayson Lynch, and Ryuhei Uehara. Unfolding and dissection of multiple cubes, tetrahedra, and doubly covered squares. J. Inf. Process., 25:610–615, 2017.



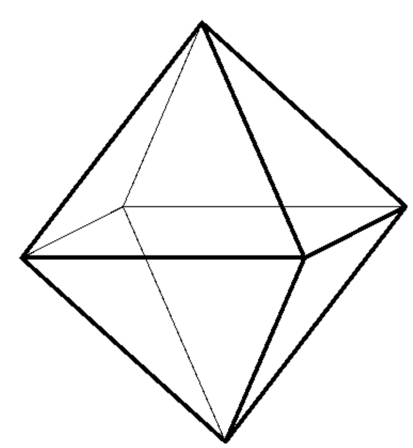
Cube



Rep-cube [1]

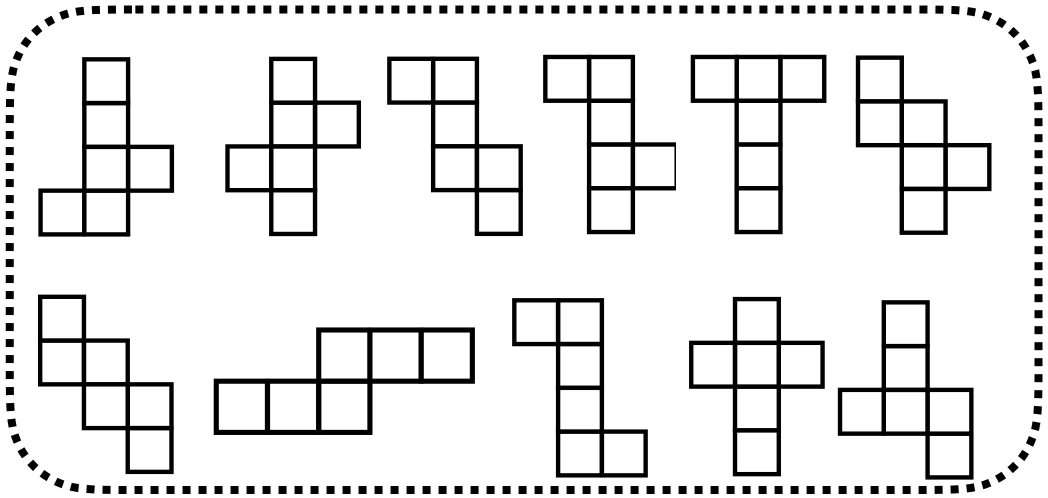
Dual

Octahedron

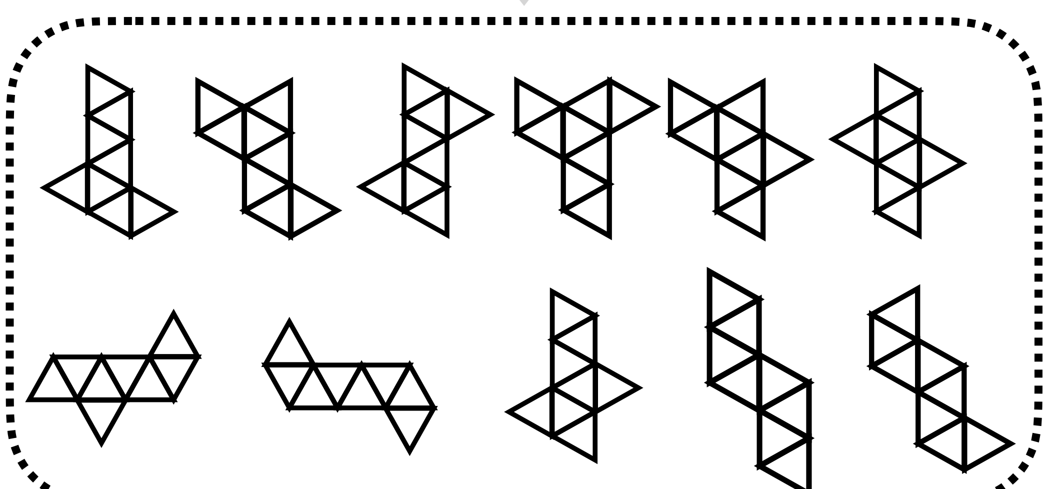


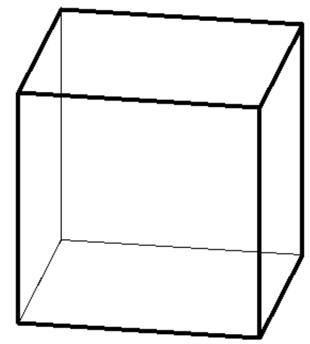
Background

[1] Zachary Abel, Brad Ballinger, Erik D. Demaine, Martin L. Demaine, Jeff Erickson, Adam Hesterberg, Hiro Ito, Irina, Kostitsyna, Jayson Lynch, and Ryuhei Uehara. Unfolding and dissection of multiple cubes, tetrahedra, and doubly covered squares. J. Inf. Process., 25:610–615, 2017.



#(Edge nets) are the same

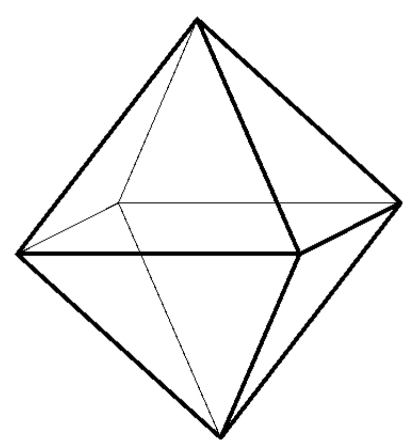


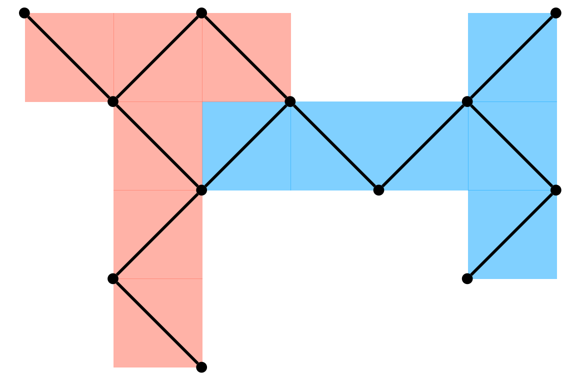


Cube

Dual

Octahedron

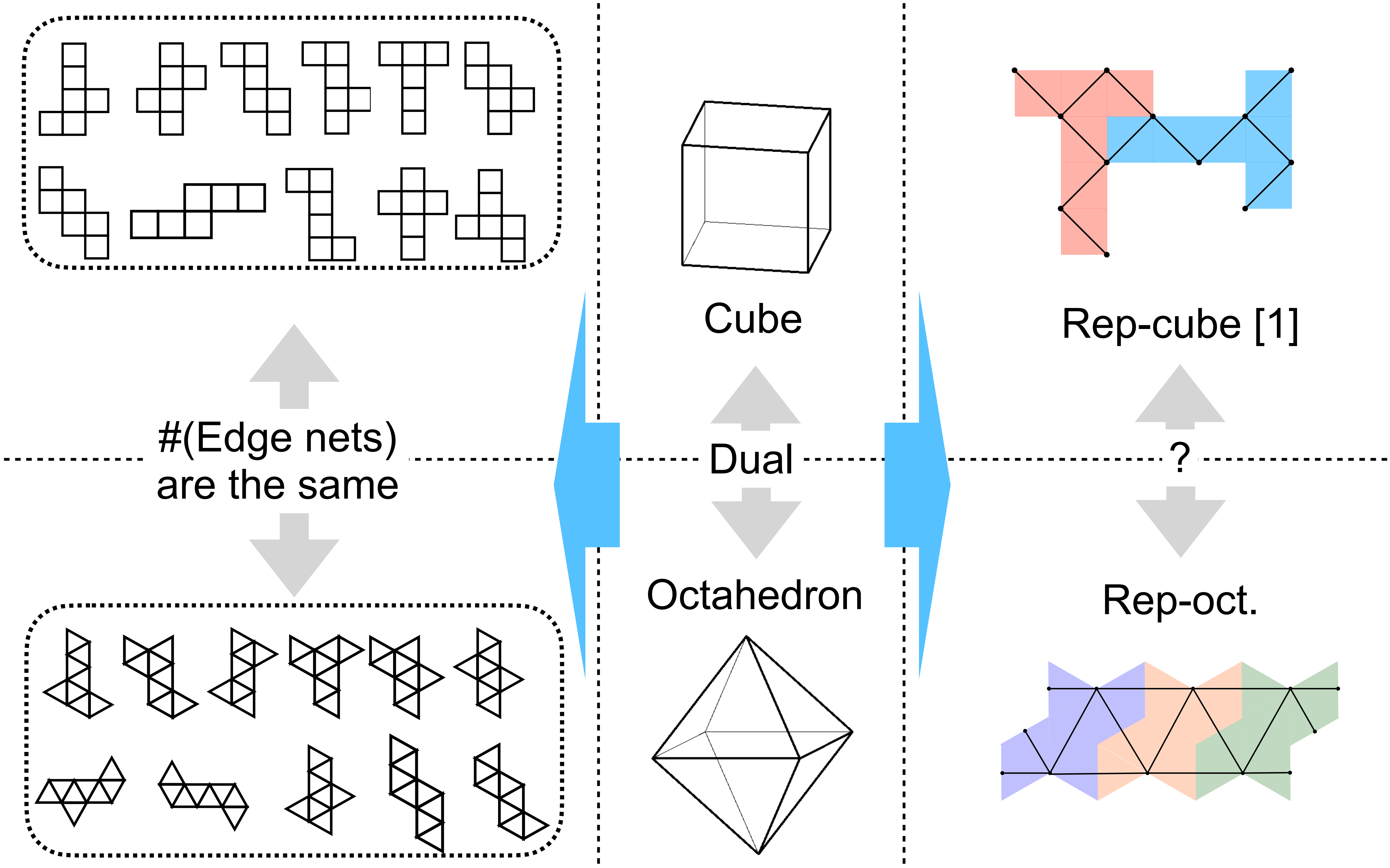




Rep-cube [1]

Background

[1] Zachary Abel, Brad Ballinger, Erik D. Demaine, Martin L. Demaine, Jeff Erickson, Adam Hesterberg, Hiro Ito, Irina, Kostitsyna, Jayson Lynch, and Ryuhei Uehara. Unfolding and dissection of multiple cubes, tetrahedra, and doubly covered squares. J. Inf. Process., 25:610–615, 2017.



Results

[(a part of) Known Results for Rep-cube] [Abel et al., 2017] [Xu et al., 2018] [Xiaoting, 2021] [Okada et al., 2022]

- As a condition the existence of a regular rep-cube with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + 2ab]$ is necessary .
  - shown that $\exists a \in \mathbb{Z}, [k = 18a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-cube exists within degrees $k \leq 8$.
  - shown that a uniform rep-cube exists for some degree k.

[Our Results for Rep-oct.]

- As a condition for the existence of a regular rep-oct. with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary.
  - shown that $\exists a \in \mathbb{Z}, [k = 64a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-oct. exists within degrees $k \leq 9$.
  - shown that sometimes no uniform rep-oct. exists for any degree k.

Results

[(a part of) Known Results for Rep-cube]

[Abel et al., 2017] [Xu et al., 2018] [Xiaoting, 2021] [Okada et al., 2022]

- As a condition the existence of a regular rep-cube with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + 2ab]$ is necessary .
  - shown that $\exists a \in \mathbb{Z}, [k = 18a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-cube exists within degrees $k \leq 8$.
  - shown that a uniform rep-cube exists for some degree k.

[Our Results for Rep-oct.]

- As a condition for the existence of a regular rep-oct. with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary.
  - shown that $\exists a \in \mathbb{Z}, [k = 64a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-oct. exists within degrees $k \leq 9$.
  - shown that sometimes no uniform rep-oct. exists for any degree k.

Results

[(a part of) Known Results for Rep-cube] [Abel et al., 2017] [Xu et al., 2018] [Xiaoting, 2021] [Okada et al., 2022]

- As a condition the existence of a regular rep-cube with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + 2ab]$ is necessary .
  - shown that $\exists a \in \mathbb{Z}, [k = 18a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-cube exists within degrees $k \leq 8$.
  - shown that a uniform rep-cube exists for some degree k.

[Our Results for Rep-oct.]

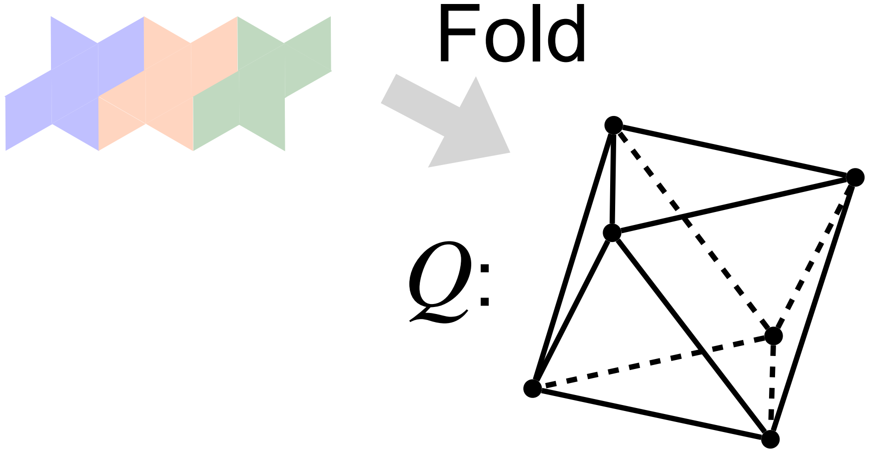
- As a condition for the existence of a regular rep-oct. with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary.
  - shown that $\exists a \in \mathbb{Z}, [k = 64a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-oct. exists within degrees $k \leq 9$.
  - shown that sometimes no uniform rep-oct. exists for any degree k.

Results 1

Theorem

$\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary to exist a regular rep-oct. with degree k.

P:



Q:

[Proof]

- Let P be a regular rep-oct. of degree k.
- Let Q be the folded octahedron from P.

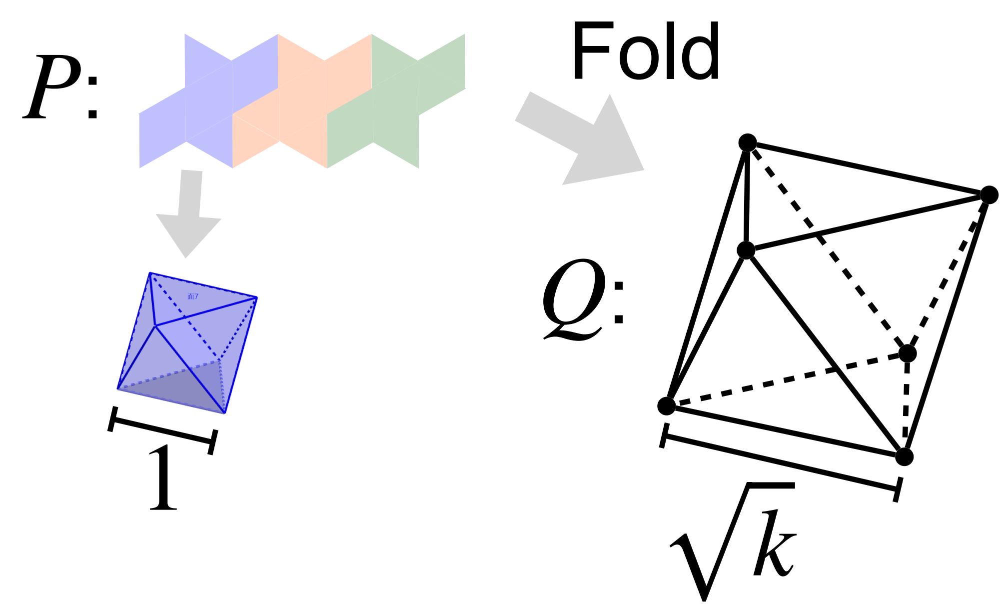
Results 1

Theorem

$\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary to exist a regular rep-oct. with degree k.

[Proof]

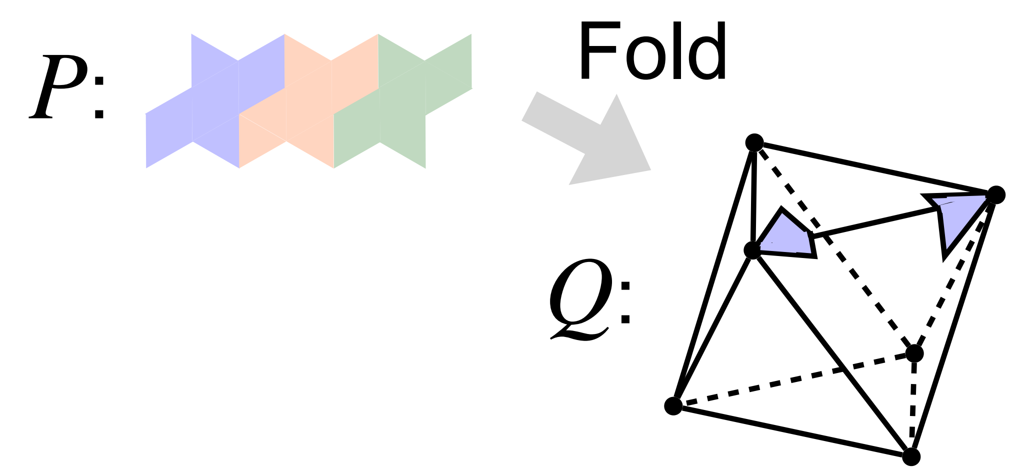
- Let P be a regular rep-oct. of degree k.
- Let Q be the folded octahedron from P.
- The length of edges of Q is $\sqrt{k}$.



Results 1

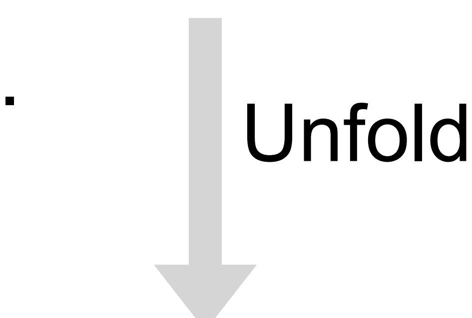
Theorem

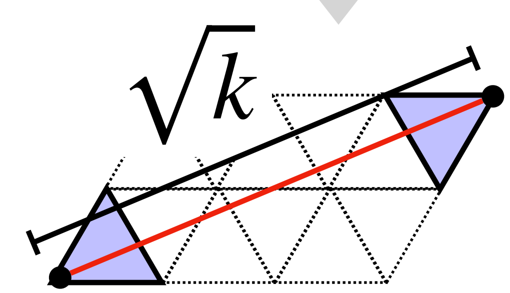
$\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary to exist a regular rep-oct. with degree k.



[Proof]

- Let P be a regular rep-oct. of degree k.
- Let Q be the folded octahedron from P.
- The length of edges of Q is $\sqrt{k}$.
- Two vertices v and v' must be points of a triangular lattice.

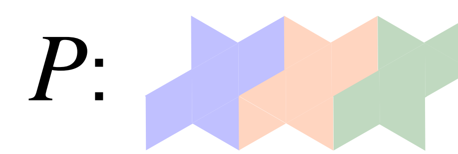


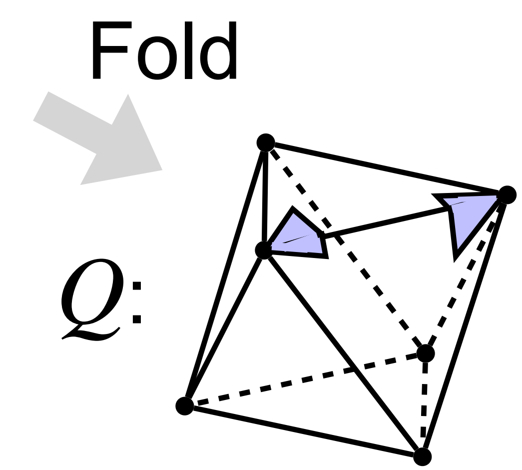


Results 1

Theorem

$\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary to exist a regular rep-oct. with degree k.

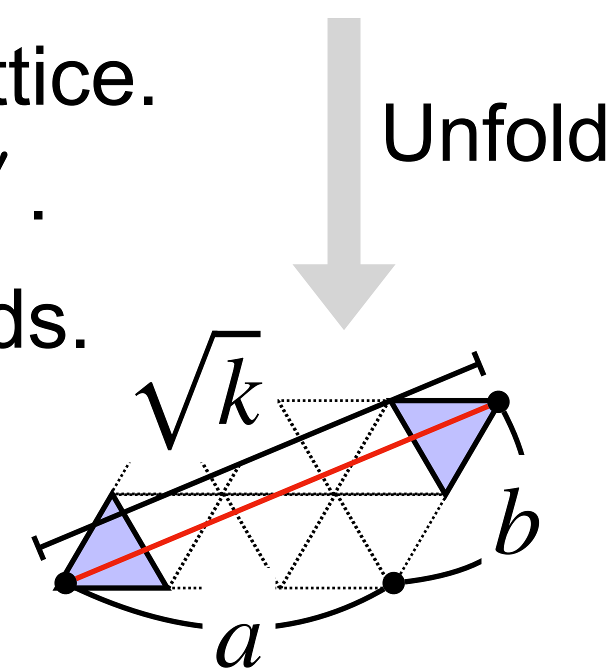




[Proof]

- Let P be a regular rep-oct. of degree k.
- Let Q be the folded octahedron from P.
- The length of edges of Q is $\sqrt{k}$.
- Two vertices v and v' must be points of a triangular lattice.
- Let (a, b) the coordinate differences between v and v'.
- It means that $(\sqrt{k})^2 = a^2 + b^2 - 2ab\cos(120°)$ holds.

$\Rightarrow k = a^2 + b^2 + ab$



Results

[(a part of) Known Results for Rep-cube] [Abel et al., 2017] [Xu et al., 2018] [Xiaoting, 2021] [Okada et al., 2022]

- As a condition the existence of a regular rep-cube with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + 2ab]$ is necessary .
  - shown that $\exists a \in \mathbb{Z}, [k = 18a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-cube exists within degrees $k \leq 8$.
  - shown that a uniform rep-cube exists for some degree k.

[Our Results for Rep-oct.]

- As a condition for the existence of a regular rep-oct. with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary.
  - shown that $\exists a \in \mathbb{Z}, [k = 64a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-oct. exists within degrees $k \leq 9$.
  - shown that sometimes no uniform rep-oct. exists for any degree k.

Results 2

Theorem

A regular rep-oct. of degree $k = 64a^2$ exists for any positive integer a.

Results 2

Theorem

A regular rep-oct. of degree $k = 64a^2$ exists for any positive integer a.

[Strategy of Proof]

Case of $a = 1$:

1. Assemble 16 edge nets into a diamond to match the notches.
2. Combine 4 diamonds and make an edge net.

Results 2

Theorem

A regular rep-oct. of degree $k = 64a^2$ exists for any positive integer a.

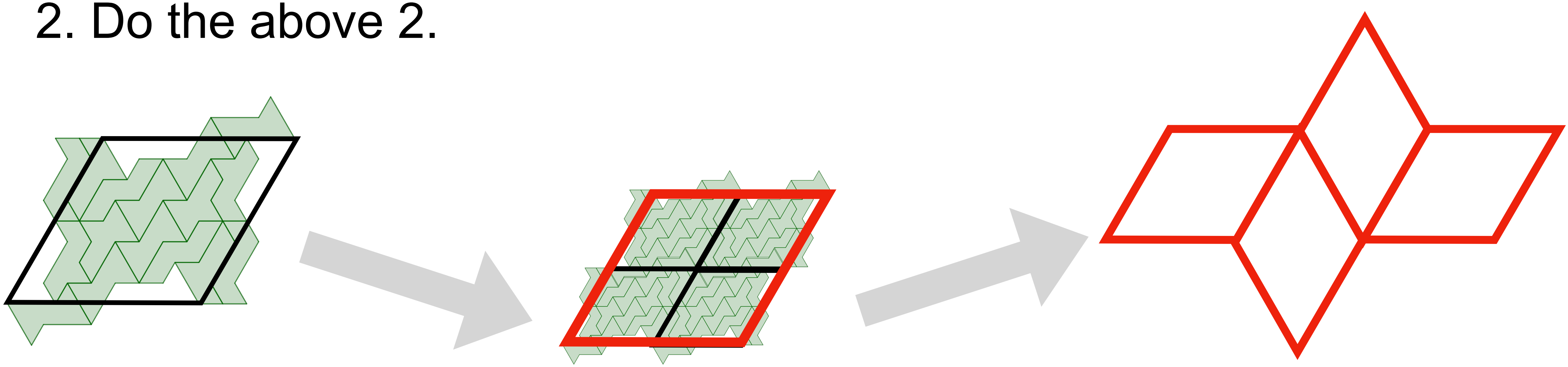
[Strategy of Proof]

Case of $a = 1$:

1. Assemble 16 edge nets into a diamond to match the notches.
2. Combine 4 diamonds and make an edge net.

Case of $a > 1$:

1. After do the above 1, combine a^2 diamonds to a bigger diamond.
2. Do the above 2.



Results 2

Theorem

A regular rep-oct. of degree $k = 64a^2$ exists for any positive integer a.

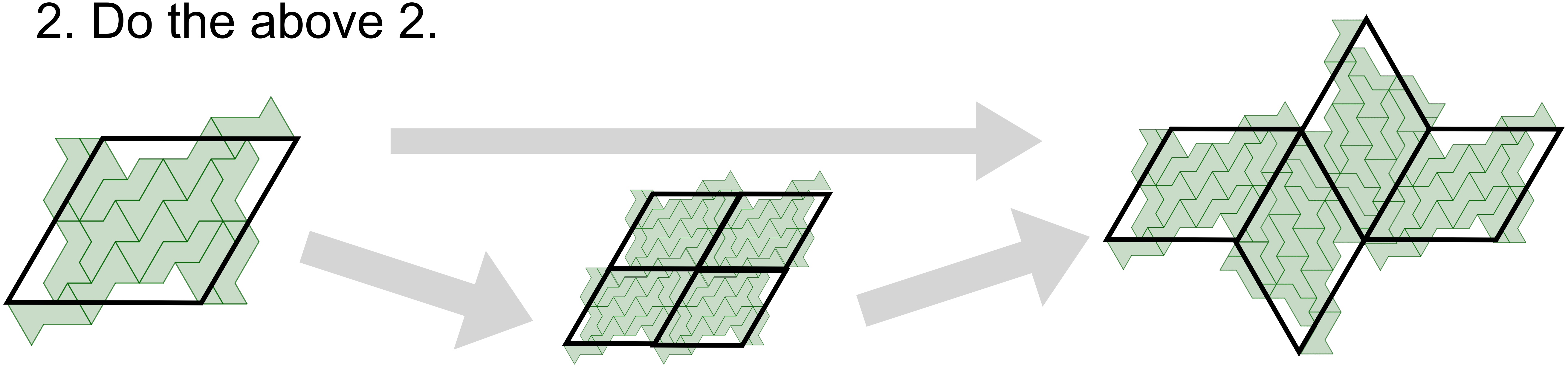
[Strategy of Proof]

Case of $a = 1$:

1. Assemble 16 edge nets into a diamond to match the notches.
2. Combine 4 diamonds and make an edge net.

Case of $a > 1$:

1. After do the above 1, combine a^2 diamonds to a bigger diamond.
2. Do the above 2.



Results

[(a part of) Known Results for Rep-cube]

[Abel et al., 2017] [Xu et al., 2018] [Xiaoting, 2021] [Okada et al., 2022]

- As a condition the existence of a regular rep-cube with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + 2ab]$ is necessary .
  - shown that $\exists a \in \mathbb{Z}, [k = 18a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-cube exists within degrees $k \leq 8$.
  - shown that a uniform rep-cube exists for some degree k.

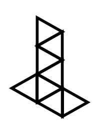
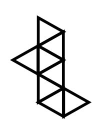
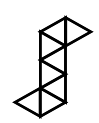
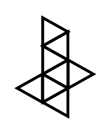
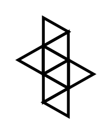
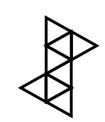
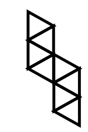
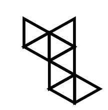
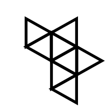
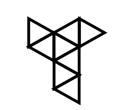
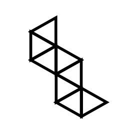
[Our Results for Rep-oct.]

- As a condition for the existence of a regular rep-oct. with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary.
  - shown that $\exists a \in \mathbb{Z}, [k = 64a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-oct. exists within degrees $k \leq 9$.
  - shown that sometimes no uniform rep-oct. exists for any degree k.

Results 3

Theorem

For each edge net of the regular oct., within degree $k \leq 9$, whether a uniform rep-oct. exists is shown in the following.

											
$k = 3$	×	○	○	×	○	×	×	○	○	×	○
$k = 4$	○	○	○	○	×	×	○	○	○	○	○
$k = 7$	×	×	×	×	×	×	×	×	×	×	×
$k = 9$	○	○	×	×	×	×	×	×	×	×	○

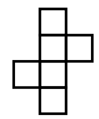
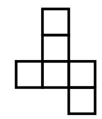
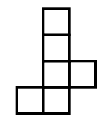
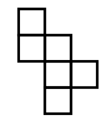
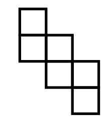
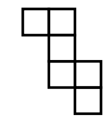
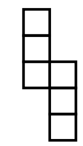
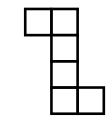
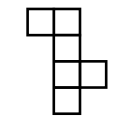
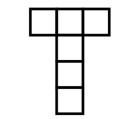
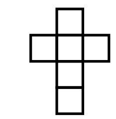
How to:
- Formularize as a integer programming problem.
- Solve it by a solver SCIP (https://www.scipopt.org/).

Results 3

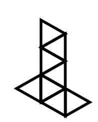
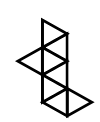
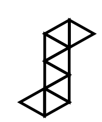
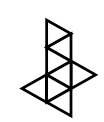
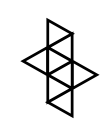
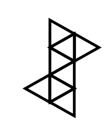
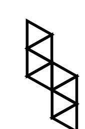
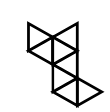
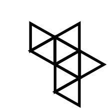
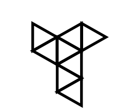
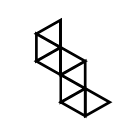
- Comparison with Rep-cube-

Rep-cube:

by [Xiaoting, 2021] and [Okada et al., 2022]

											
$k=2$	×	○	○	×	×	×	○	×	×	○	×
$k=4$	○	○	○	○	×	○	×	×	○	○	○
$k=5$	○	○	○	○	×	○	×	×	○	×	○
$k=8$	○	○	○	○	○	○	○	○	○	○	○

Rep-oct.:

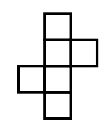
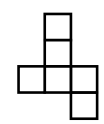
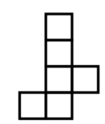
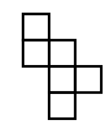
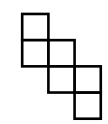
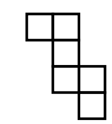
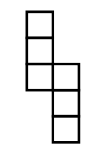
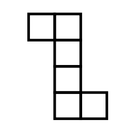
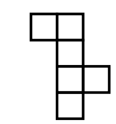
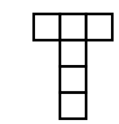
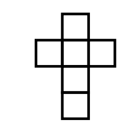
											
$k=3$	×	○	○	×	○	×	×	○	○	×	○
$k=4$	○	○	○	○	×	×	○	○	○	○	○
$k=7$	×	×	×	×	×	×	×	×	×	×	×
$k=9$	○	○	×	×	×	×	×	×	×	×	○

Results 3

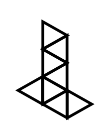
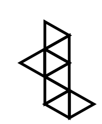
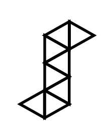
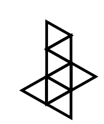
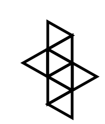
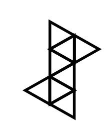
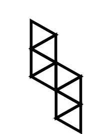
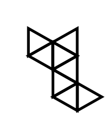
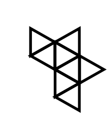
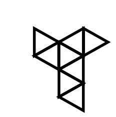
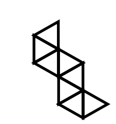
- Comparison with Rep-cube-

Rep-cube:

by [Xiaoting, 2021] and [Okada et al., 2022]

											
$k=2$	×	○	○	×	×	×	○	×	×	○	×
$k=4$	○	○	○	○	×	○	×	×	○	○	○
$k=5$	○	○	○	○	×	○	×	×	○	×	○
$k=8$	○	○	○	○	○	○	○	○	○	○	○

Rep-oct.:

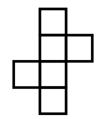
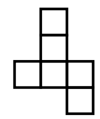
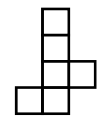
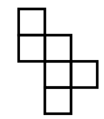
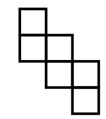
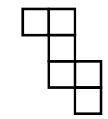
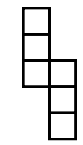
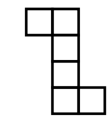
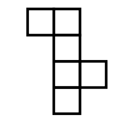
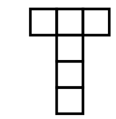
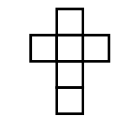
											
$k=3$	×	○	○	×	○	×	×	○	○	×	○
$k=4$	○	○	○	○	×	×	○	○	○	○	○
$k=7$	×	×	×	×	×	×	×	×	×	×	×
$k=9$	○	○	×	×	×	×	×	×	×	×	○

Results 3

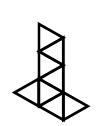
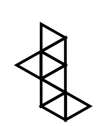
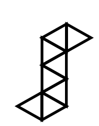
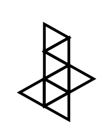
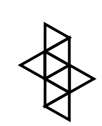
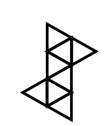
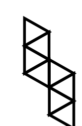
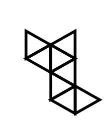
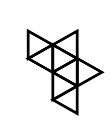
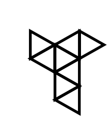
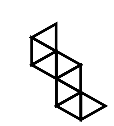
- Comparison with Rep-cube-

Rep-cube:

by [Xiaoting, 2021]
and
[Okada et al., 2022]

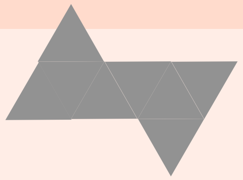
											
$k=2$	×	○	○	×	×	×	○	×	×	○	×
$k=4$	○	○	○	○	×	○	×	×	○	○	○
$k=5$	○	○	○	○	×	○	×	×	○	×	○
$k=8$	○	○	○	○	○	○	○	○	○	○	○

Rep-oct.:

											
$k=3$	×	○	○	×	○	×	×	○	○	×	○
$k=4$	○	○	○	○	×	×	○	○	○	○	○
$k=7$	×	×	×	×	×	×	×	×	×	×	×
$k=9$	○	○	×	×	×	×	×	×	×	×	○

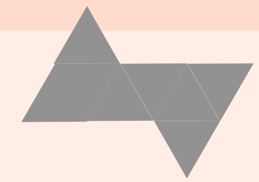
Results 4

Theorem

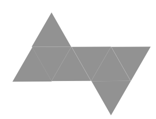
There is no uniform rep-oct. based on  .

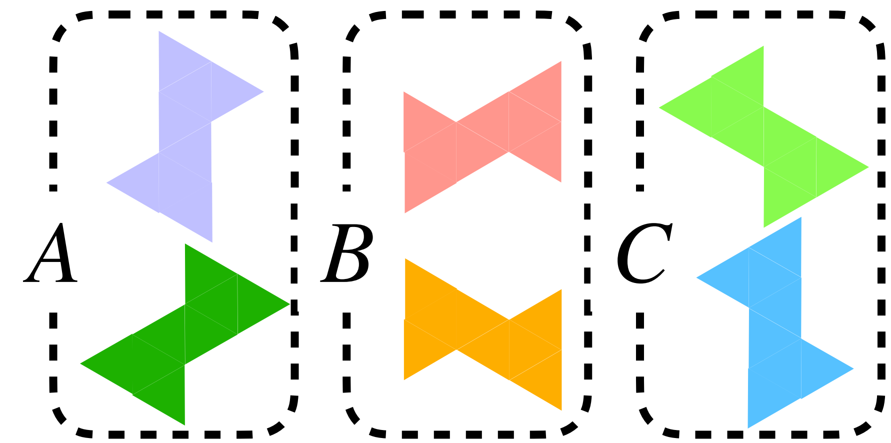
Results 4

Theorem

There is no uniform rep-oct. based on  .

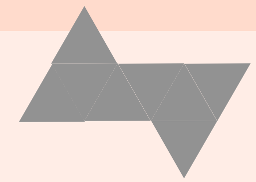
[Strategy of Proof]

- Assume that there exists uniform rep-oct. P.
- Classify ways of placing  as types A, B, C.

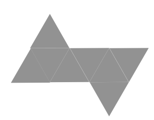


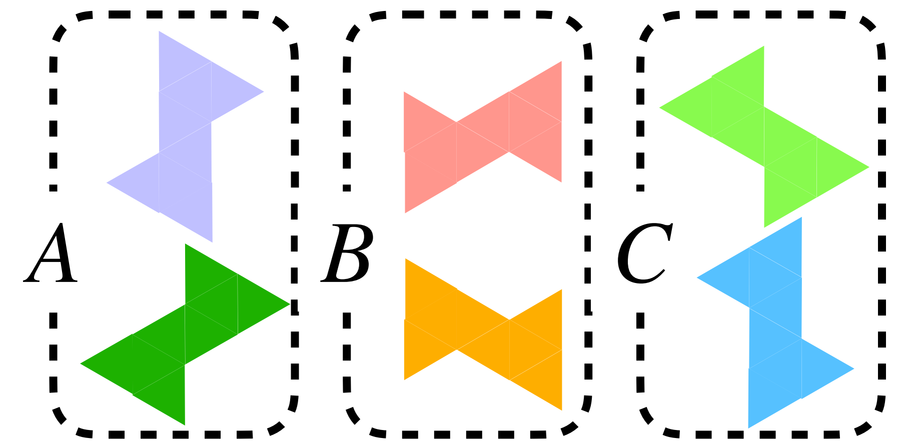
e

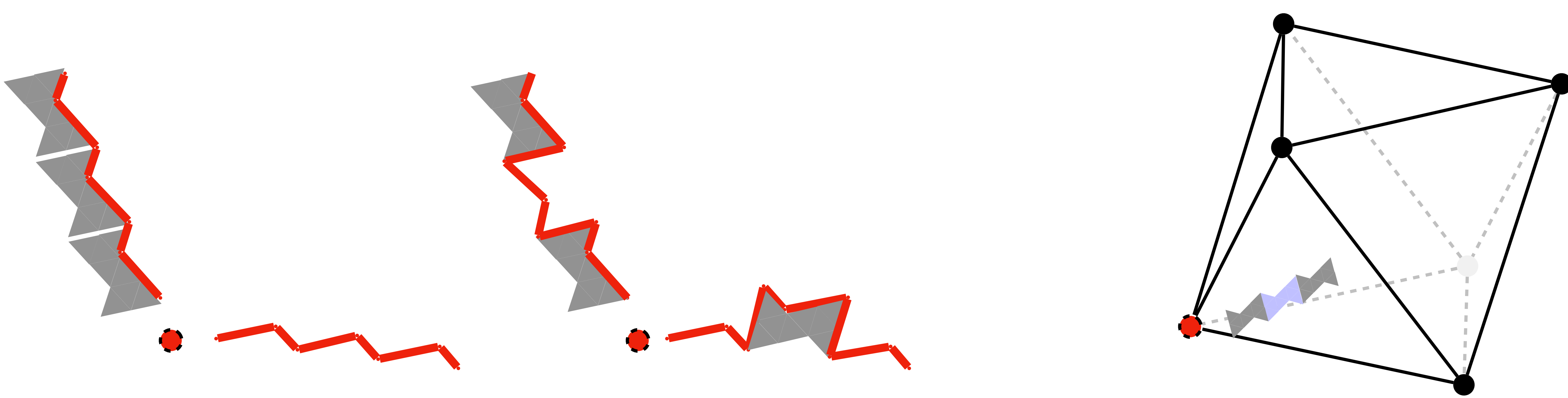
Theorem

There is no uniform rep-oct. based on .

[Strategy of Proof]

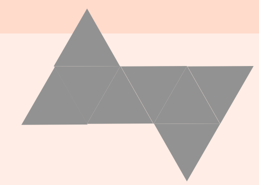
- Assume that there exists uniform rep-oct. P.
- Classify ways of placing  as types A, B, C.
- Assume that P consists of single type.
  - Show we can modify P s.t. it contains multiple types.



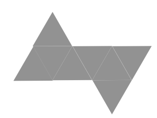


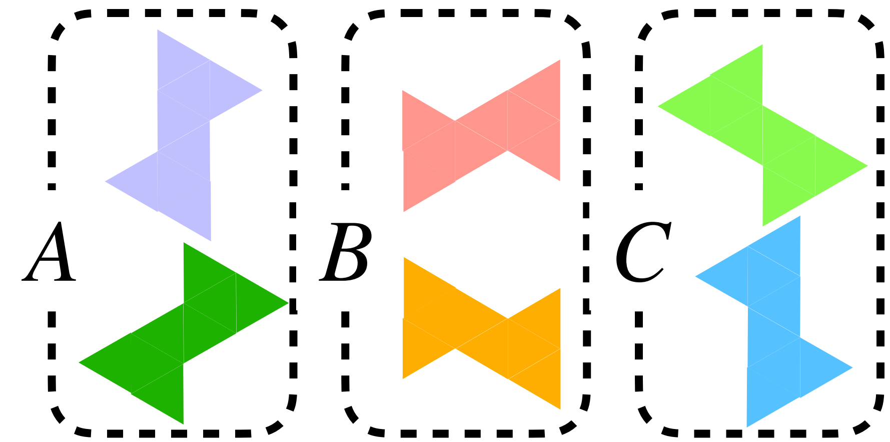
Results 4

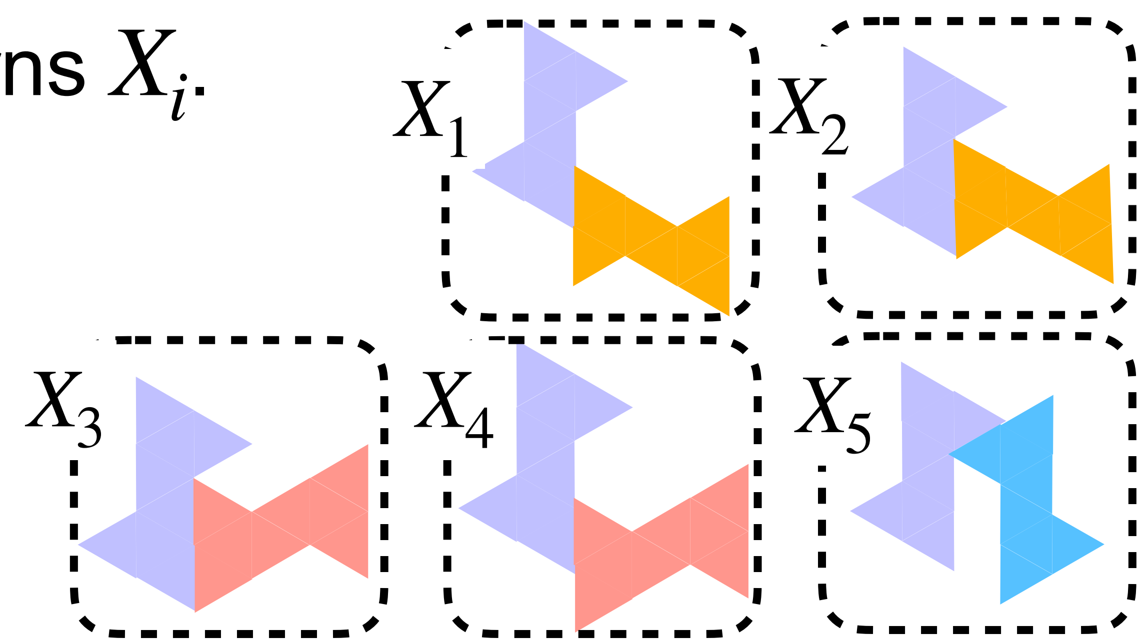
Theorem

There is no uniform rep-oct. based on  .

[Strategy of Proof]

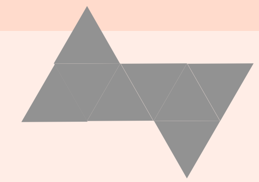
- Assume that there exists uniform rep-oct. P.
- Classify ways of placing  as types A, B, C.
- Assume that P consists of single type.
  - Show we can modify P s.t. it contains multiple types.
- Show that, if P contains multiple types, then it contradict.
  - If so, P must contain one of the patterns X_i.



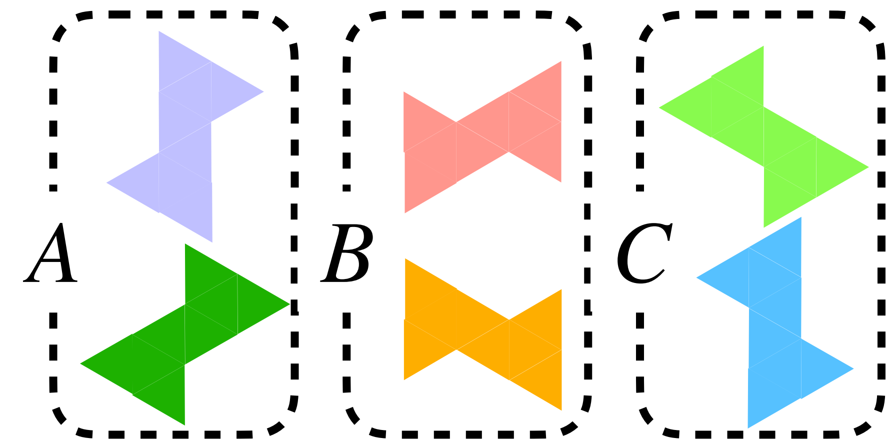


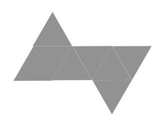
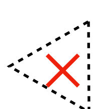
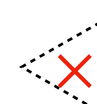
Results 4

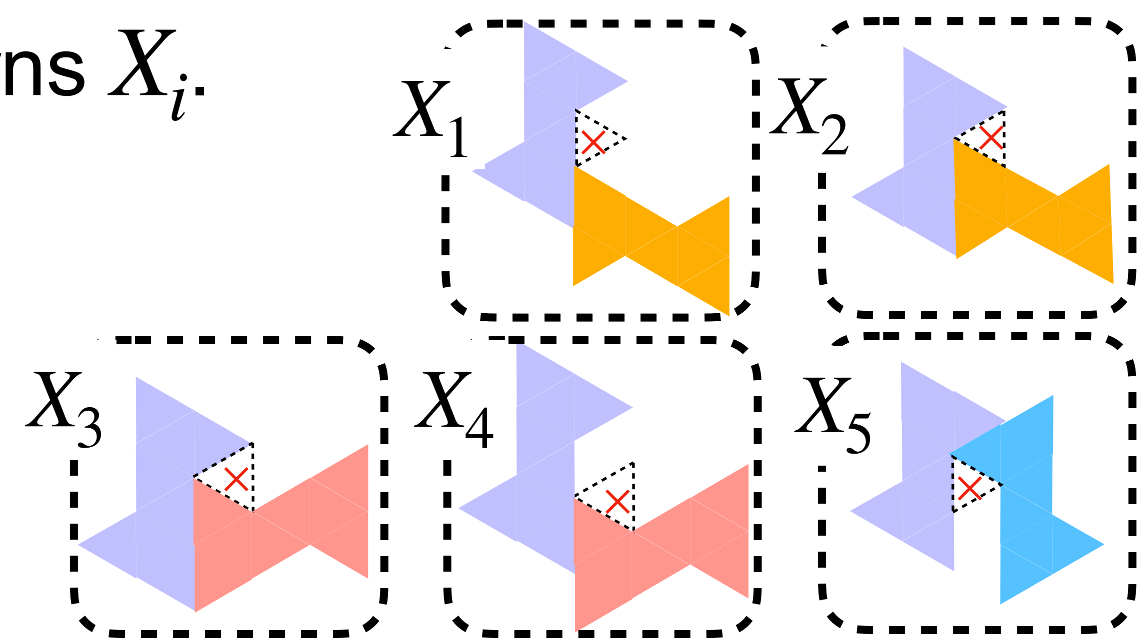
Theorem

There is no uniform rep-oct. based on .

[Strategy of Proof]



- Assume that there exists uniform rep-oct. P.
- Classify ways of placing  as types A, B, C.
- Assume that P consists of single type.
  - Show we can modify P s.t. it contains multiple types.
- Show that, if P contains multiple types, then it contradict.
  - If so, P must contain one of the patterns X_i.
  -  must be filled by another piece.
  - but it generate an overlap or new .
  - It is a contradiction.



Results

[(a part of) Known Results for Rep-cube] [Abel et al., 2017] [Xu et al., 2018] [Xiaoting, 2021] [Okada et al., 2022]

- As a condition the existence of a regular rep-cube with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + 2ab]$ is necessary .
  - shown that $\exists a \in \mathbb{Z}, [k = 18a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-cube exists within degrees $k \leq 8$.
  - shown that a uniform rep-cube exists for some degree k.

[Our Results for Rep-oct.]

- As a condition for the existence of a regular rep-oct. with degree k,
  - shown that $\exists a, b \in \mathbb{Z}, [k = a^2 + b^2 + ab]$ is necessary.
  - shown that $\exists a \in \mathbb{Z}, [k = 64a^2]$ is sufficient.
- For each 11 edge nets,
  - determined whether a uniform rep-oct. exists within degrees $k \leq 9$.
  - shown that sometimes no uniform rep-oct. exists for any degree k.