Classification and Visualization of Polyhedral Graphs via Degree Sequences

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1 Introduction

A polyhedral graph is a simple graph that is 3-connected and planar. This class of graphs corresponds exactly to the vertex-edge graphs (or 1-skeletons) of convex polyhedra, as stated in the following classical theorem.

Theorem 1 (Steinitz's theorem [9, 28]). *A graph is the 1-skeleton of a convex polyhedron if and only if it is simple, planar, and 3-connected.*

In graph theory, classifying graphs is a standard way to understand their structural properties. Convex regular-faced polyhedra—such as the Platonic, Archimedean, and Johnson solids—are one of the most well-studied classes of polyhedral graphs [5, 14]. These polyhedral graphs are typically classified according to geometric features such as face regularity and global symmetry. In addition, polyhedral graphs can also be classified based on combinatorial parameters such as the number of edges or vertices. Duijvestijn and Federico enumerated the number of non-isomorphic polyhedral graphs with 6 to 26 edges [7, 6]. The numbers of non-isomorphic polyhedral graphs with 4 to 18 vertices are also known [27]. More recently, Maffucci has proposed a classification approach for polyhedral graphs based on their degree distributions—that is, how many vertices have each degree in the graph. In 2022, he showed—via degree distribution analysis—that there are exactly three polyhedral graphs whose complements are also polyhedral [18]. In 2024, he showed that only eight degree distributions are forcibly polyhedral [19]. Maffucci's results suggest that degree distributions provide a useful basis for classifying polyhedral graphs. Polyhedral graphs have thus been classified in a variety of ways.

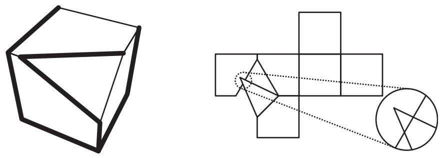

Figure 1: A cube with truncated corners and its overlapping unfolding [21].

Edge unfoldings are structure closely associated with polyhedra. Here, an edge unfolding is a flat polygon obtained by cutting along the polyhedron's edges and unfolding the surface onto a plane. Depending on the shape of the polyhedron and how it is edge unfolded, the resulting unfolding may have overlaps, i.e., two distinct faces overlap, or their boundaries are in touch (see Figure 1). Shephard proposed the following conjecture about edge unfoldings.

Conjecture 2 ([25]). *For any convex polyhedron, at least one non-overlapping edge unfolding exists.*

This conjecture is called Dürer's problem. The name comes from a sketch in "Underweysung der Messung mit dem Zirckel und Richtscheyt" [8], written by Albrecht Dürer in 1525, which is often regarded as the origin of edge unfoldings [5]. One approach to this problem is to develop an edge unfolding algorithm that works for all convex polyhedra. Schlickenrieder proposed an edge unfolding algorithm that successfully applies to $60,000$ randomly generated convex polyhedra [24]. However, Lucier showed that there exists a polyhedron that cannot be unfolded without overlap using Schlickenrieder's proposed algorithm [17]. We note that this polyhedron can still be unfolded without overlap by a different method.

Accordingly, edge unfolding algorithms are now being developed for specific families rather than for

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general convex polyhedra. In particular, prismatoids have received increasing attention in recent years [2, 3, 22, 23]. Here, prismatoids are convex polyhedra whose vertices lie on two parallel planes; this class includes prisms and pyramids. These works introduce structural constraints on prismatoids—for example, relations between top and bottom faces—and propose algorithms for each case. This focus is likely due to the structural simplicity of prismatoids. Each prismatoid is determined by its top and bottom faces—typically an t-gon and an b-gon—so varying t and b naturally yields a wide range of instances. In contrast, few families are known—beyond prismatoids—that support systematic variation of parameters such as t or b. Accordingly, the development of edge unfolding algorithms for these more general classes remains limited.

Our contributions We propose a new method for classifying polyhedral graphs based on their degree distributions, following the idea introduced by Maffucci. This approach is motivated by the lack of known parameterized families among polyhedra beyond prismatoids. To this end, we define the degree distribution of a graph with n vertices as follows:

$$d_1^{c_1}, d_2^{c_2}, \dots, d_k^{c_k} \quad \text{where} \quad \sum_{i=1}^{k} c_i = n. \tag{1}$$

Here, each term $d_i^{c_i}$ indicates that the graph has c_i vertices of degree d_i. In this study, we specifically focus on polyhedral graphs with the following degree distributions.

$$\begin{aligned} &d_1^{n-x}, (n-y)^{c_2}, d_3^{c_3}, \dots, d_k^{c_k} \\ &\text{where} \quad 0 \le x, y \le n,\ x, y \in \mathbb{N},\ \sum_{i=2}^{k} c_i = x. \end{aligned} \tag{2}$$

Figures 2,3, and4 illustrate example families of polyhedral graphs that follow this degree distribution form. The graphs are visualized to make their structure easier to see. Note that, for each n, the graphs shown are the only non-isomorphic polyhedral graphs with this degree distribution.

We further organize the families by introducing directed edges that represent structural transitions across increasing values of n. This relation is defined by the following steps (see Figure 5):

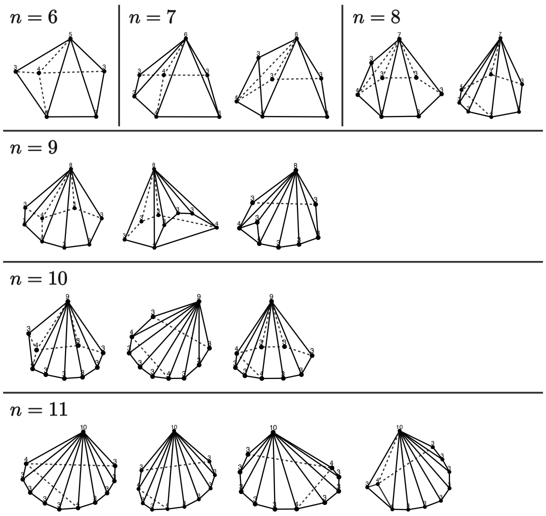

Figure 2: Polyhedral graphs with degree distribution $3^{n-3}, (n-1)^1, 4^2$.

Step 1. Given a polyhedral graph with n vertices, add a new vertex of degree d_1 and connect it to the graph.

Step 2. If the resulting graph is a polyhedral graph with $n+1$ vertices in the same degree distribution form, draw a directed edge from the n-vertex graph to the $n+1$-vertex graph.

Note that when a new vertex is added, if the resulting face becomes flat (i.e., lies in a plane), the corresponding diagonal edge is removed from the graph. Figures 6,7, and 8 show example directed graphs formed by connecting polyhedral graphs according to the above procedure.

In this way, our approach provides a unified framework for classifying polyhedral graphs based on their degree distributions and organizing them into structured families. By visualizing their structural transitions, we clarify how polyhedral families evolve as the number of vertices increases. This perspective offers a foundation for future studies of edge unfolding algorithms, including implications for future investigations into Dürer's problem.

Structure of the paper This paper is organized as follows. Section 2 introduces basic definitions and notation. Section 3 outlines the motivation behind classifying polyhedral graphs, including our

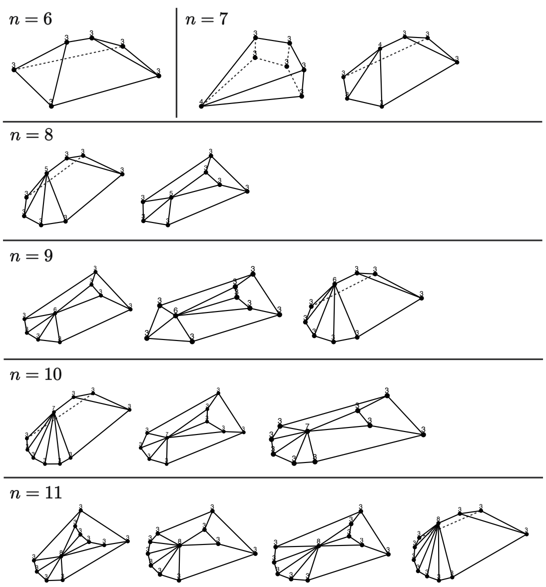

Figure 3: Polyhedral graphs with degree distribution $3^{n-1}, (n-3)^1$.

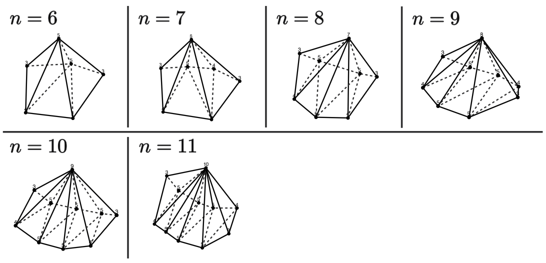

Figure 4: Polyhedral graphs with degree distribution $5^{n-5}, (n-1)^1, 3^2, 4^2$.

ideas related to Dürer's problem that led to this direction. Section 4 describes the method for extracting polyhedral graphs that conform to the degree distribution structure given in equation (2). Finally, Section 5 concludes the paper with a summary and remarks on future directions.

2 Preliminaries

Let $G = (V, E)$ be a simple graph where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A graph is said to be *planar* if it can be embedded in the plane without any edge crossings, that is, if it can be drawn such that no two edges intersect except at their endpoints. A sequence of vertices $\langle v_1, \ldots, v_k \rangle$ is a *path* if $v_i \neq v_j$ ($v_i,\ v_j \in V,\ 1 \leq i \neq j \leq k$) and every consecutive two vertices are adjacent. A graph is *connected* if a path exists between any two vertices of the graph. A connected graph G is *k-vertex-connected* (or simply *k-connected*) if $k < |V|$ and the graph remains connected after the removal of any set of fewer than k vertices. A graph that is simple, planar, and 3-connected is called a *polyhedral graph*.

A *polyhedron* is a three-dimensional object consisting of at least four polygons, called *faces*, joined at their edges. A *convex polyhedron* is a polyhedron in which the dihedral angle between any two adjacent faces is strictly less than π. According to Steinitz's theorem (Theorem 1), every polyhedral graph corresponds to the 1-skeleton of a convex polyhedron, and vice versa. In what follows, we use the terms polyhedron and polyhedral graph interchangeably unless otherwise noted; for instance, we may refer to a prism graph simply as a prism.

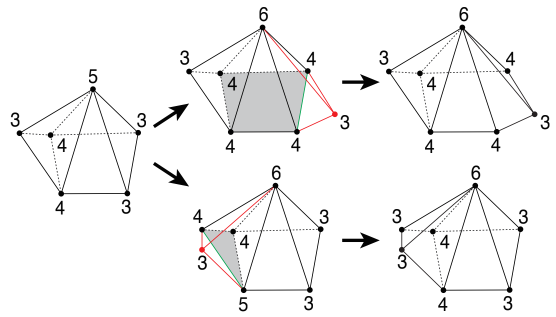

Figure 5: A vertex of degree 3 is added to the $n = 6$ polyhedral graph with degree distribution $3^{n-3}, (n-1)^1, 4^2$ (see Figure 2). When the red vertex lies in the same plane as the gray face, the green diagonal is removed. The resulting graphs correspond to the $n = 7$ polyhedral graphs with the same degree distribution form.

3 Motivation for classifying polyhedral graphs

In this section, we first present our idea related to Dürer's problem, which motivates our classification of polyhedral graphs. We then explain why

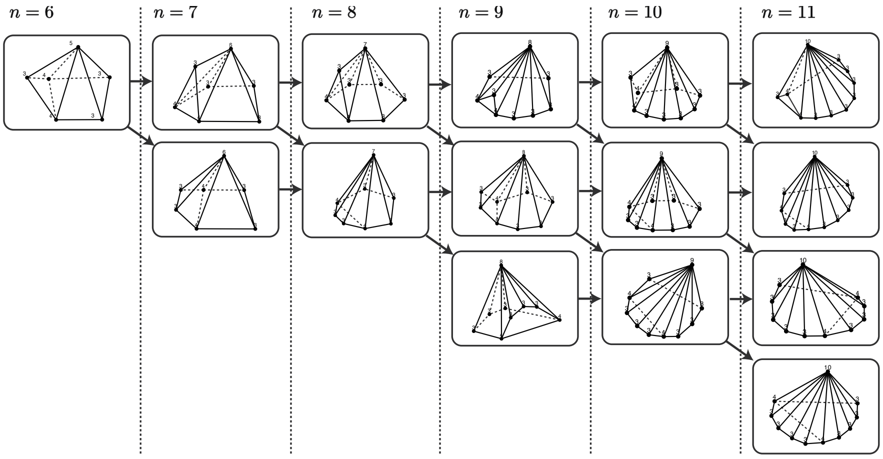

Figure 6: Organized structure of the polyhedral graphs shown in Figure 2, with degree distribution $3^{n-3}, (n-1)^1, 4^2$.

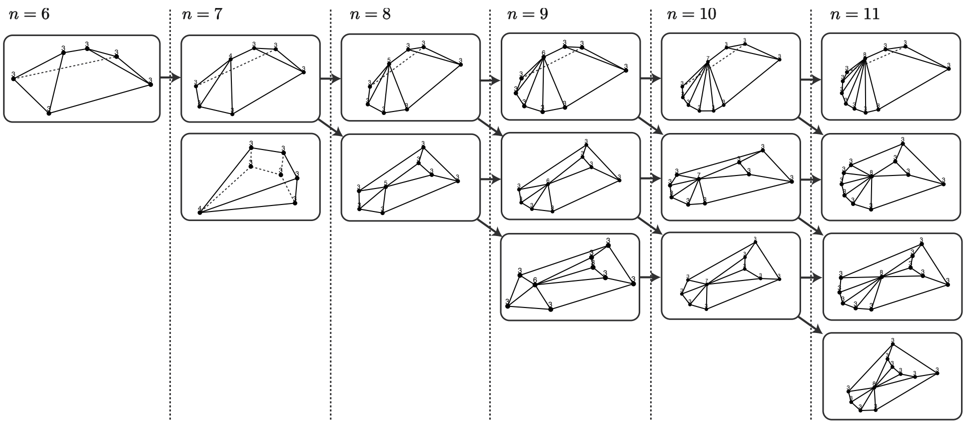

Figure 7: Organized structure of the polyhedral graphs shown in Figure 3, with degree distribution $3^{n-1}, (n-3)^1$.

we adopted the degree distribution defined in equation (2) as the basis for this classification.

3.1 Our approach to Dürer's problem

A natural approach to Dürer's problem is to develop an edge unfolding algorithm that can be applied to all convex polyhedra. However, since convex polyhedra can vary greatly in size and structure, it is difficult to develop a single algorithm that works directly in all cases. To overcome this challenge, we propose a divide-and-combine approach to edge unfolding, as follows (see Figure 9).

Step 1. Divide the given polyhedron into several *parts*, each consisting of a connected set of faces.

Step 2. Apply an edge unfolding algorithm to each part so that there are no overlaps.

Step 3. Combine the unfolded parts, and then form a single edge unfolding of the original polyhedron.

One key challenge in this approach is managing the trade-off between unfolding and combining. As shown in Table 1, if the number of parts is increased, it becomes easier to unfold each part with-

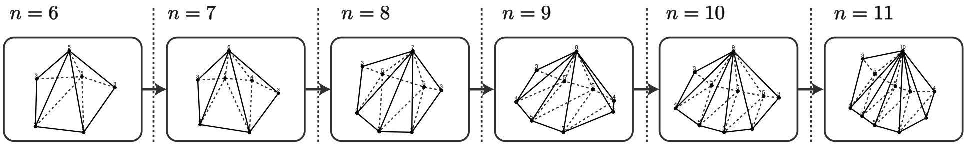

Figure 8: Organized structure of the polyhedral graphs shown in Figure 3, with degree distribution $5^{n-5}, (n-1)^1, 3^2, 4^2$.

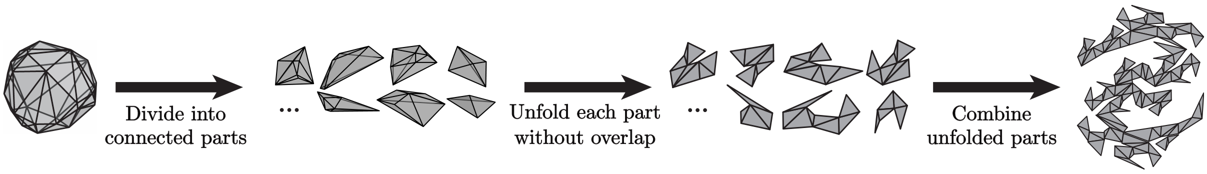

Figure 9: Overview of our divide-and-combine approach to edge unfolding.

out overlap, but combining them into a single unfolding becomes significantly more difficult. Conversely, if the number of parts is reduced, combining them becomes somewhat easier, but the variety of parts increases rapidly, and a non-overlapping unfolding algorithm is needed for each. To address this trade-off, we propose two strategies for effectively dividing a polyhedron within our divide-and-combine approach.

Strategy 1. The first strategy is to select edge-overlap-free parts. A polyhedron is said to be *edge-overlap-free* if none of its edge unfoldings results in overlaps [16]. By definition, such parts can be unfolded without overlap, regardless of how their faces are connected. This flexibility in rearranging faces makes the combination step easier to handle within our divide-and-combine approach.

Edge-overlap-freeness has been shown for several types of polyhedra. For example, tetramonohedra —tetrahedra with all faces being congruent triangles—have been shown to be edge-overlap-free [1]. Edge-overlap-freeness has also been fully characterized for convex regular-faced polyhedra whose edges all have the same length [4, 10, 12, 13, 26]. More recently, it has been proven that regular k-gonal prisms for $3 \leq k \leq 10$ are edge-overlap-free for any height [15].

Strategy 2. The second strategy is to select parts that can be unfolded without overlap in multiple ways. For instance, regular k-gonal prisms can be unfolded in a variety of non-overlapping ways, such as those shown in Figure 10, regardless of the values of k and height h. Having multiple non-overlapping unfoldings allows for greater flexibility in combining parts, as different variations can be explored. Although this condition is weaker than edge-overlap-freeness, the ability to rearrange faces in multiple ways can still make the combination step easier within our divide-and-combine approach.

To make these strategies effective, the parts need to be sufficiently large. In particular, if a polyhedron can be divided into large parts that satisfy either of the two strategies above, the combination step can be made much easier. However, it has remained unclear what kinds of parts are actually suitable for such strategies. This observation led us to focus on classifying polyhedral graphs.

3.2 Classifying polyhedral graphs by degree distribution

Our classification is based on the structure of the cut surface that appears when a part is removed from a polyhedron. This surface typically forms an k-gon, as illustrated in Figure 11. Note that the resulting k-gon is not always planar. For instance, in Figure 11, the cut surface in the top example lies on a plane, whereas the one in the bottom example does not.

Polyhedra with a k-gonal base of this kind include well-known examples such as pyramids,

Table 1: Trade-offs in the number of divisions in our edge unfolding algorithm. Bold entries indicate the more favorable option in each row.

Number of divisions	Few	Many
Number of unfolding algorithms required	Very many	**Relatively few**
Difficulty of designing non-overlapping unfoldings	Very high	**Relatively low**
Difficulty of combining parts	**Relatively low**	Very high

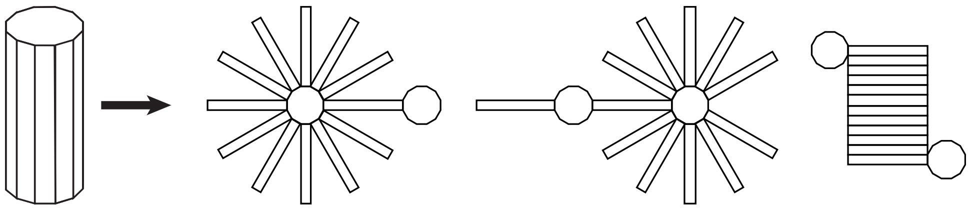

Figure 10: Examples of edge unfolding variations for regular k-gonal prisms that avoid overlap regardless of k and height h.

Table 2: Degree distributions of pyramids.

$\|V\|$ \ deg	3	4	5	6	7	8	9	10
4	4							
5	4	1						
6	5		1					
7	6			1				
8	7				1			
9	8					1		
10	9						1	
11	10							1

Table 3: Degree distributions of prisms and antiprisms.

Prisms

$\|V\|$ \ deg	3
6	6
8	8
10	10
12	12

Antiprisms

$\|V\|$ \ deg	4
6	6
8	8
10	10
12	12

prisms, and antiprisms (see Figure 12). Other polyhedra, such as prismatoids and their stacked variants, can also be viewed as parts based on a k-gonal base (see Figure 13).

Each of these families forms a continuous sequence of structures, where the shape changes regularly as the number of vertices increases. To investigate such regular structural transitions, we turn our attention to the degree distribution of polyhedral graphs, inspired by an idea of Maffucci. As an initial observation, we checked the degree distributions of pyramids, prisms, and antiprisms, and found that they follow a consistent pattern (see Tables 2 and 3). These degree distributions can be written in the form of equation (1) as follows.

$$\begin{aligned}&\text{Pyramid:} && 3^{n-1}, (n-1)^1\\ &\text{Prism:} && 3^n\\ &\text{Antiprism:} && 4^n\end{aligned}$$

Based on this observation, we use the following form as a base structure:

$$d_1^{n-x}, (n-y)^{c_2}, \tag{3}$$

where x and y are fixed integers with $0 \leq x, y \leq n$. Note that for prisms and antiprisms, this reduces to the case $y = n$. In contrast, many polyhedra include degree values that appear with a fixed count, independent of n. To capture such n-independent substructures, we extend equation (3) by appending constant pairs $d_i^{c_i}$, leading to the general form in equation (2).

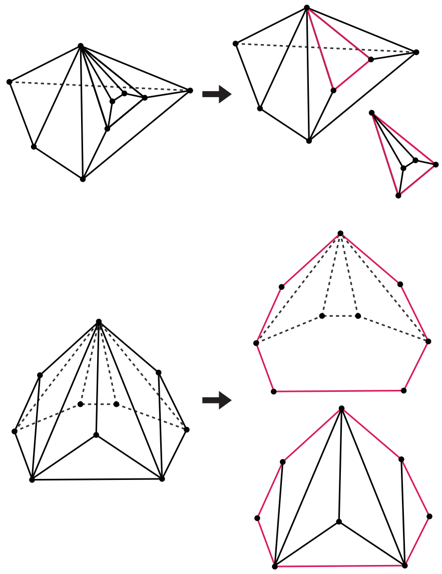

Figure 11: A polyhedron (left) and the resulting shape after removing one part (right). The red edges indicate the cut boundary, which forms an k-gon in each case.

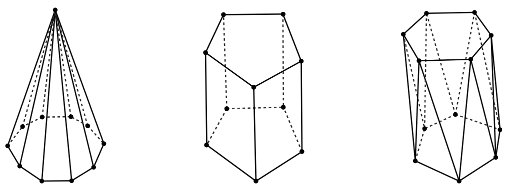

Figure 12: Examples of polyhedra with an k-gonal base.

4 Classification of polyhedral graphs

In this section, we describe how to extract families of polyhedral graphs whose degree distributions follow the structure given in equation 2. The extraction consists of the following three steps:

Step 1. Enumerate all non-isomorphic polyhedral graphs, excluding rotational and reflective symmetries.

Step 2. For each vertex count n, determine the degree distributions that appear among the enumerated graphs.

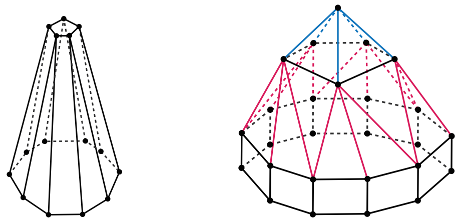

Figure 13: Examples of prismatoids and their stacked combinations. The right figure shows a structure composed of three stacked prismatoids, corresponding to the blue, red, and black parts.

Step 3. For each n, compare the degree distributions with those for $n+1$, $n+2$, and so on, and identify families that follow the structure in equation 2.

In Step 1, we use the program `nauty` [20] to enumerate only non-isomorphic polyhedral graphs. All computational experiments were conducted on an M3 chip with 24GB of memory, running macOS 15.5.

Through the above extraction process, we identified a variety of sequential families of polyhedral graphs. Among these, we focus on five representative families and provide visualizations for the following:

- $3^{n-3}, (n-1)^1, 4^2 \quad (n \geq 6,$ Figure 6$)$
- $3^{n-1}, (n-3)^1 \quad (n \geq 6,$ Figure 7$)$
- $5^{n-5}, (n-1)^1, 3^2, 4^2 \quad (n \geq 6,$ Figure 8$)$

5 Conclusion

We proposed a new method for classifying polyhedral graphs by their degree distributions. This classification is motivated by our broader goal of developing edge unfolding algorithms as a step toward resolving Dürer's problem. This classification allows us to further investigate edge-overlap-freeness and to develop algorithms for generating non-overlapping edge unfoldings of polyhedra beyond prismatoids.

There remain several directions for future work. First, while this paper visualized only a limited set of families, many other degree distributions can also be expressed in the form of equation (2). We plan to visualize these additional cases and reveal their structural relationships, with the aim of identifying further families. Second, the case of vertex counts $n \geq 12$ has not yet been investigated. While we used `nauty` to enumerate non-isomorphic graphs, the computation did not finish within 24 hours for $n = 12$. However, since polyhedral graphs are planar, isomorphism testing can be performed in polynomial time [11]. If enumeration becomes feasible for $n \geq 12$, we may be able to identify additional families that extend in increments of two, such as $n \to n+2 \to n+4 \to \dots$. Third, we consider the task of determining how many polyhedral graphs correspond to a given degree distribution. For instance, we observed that the distribution $5^{n-5}, (n-1)^1, 3^2, 4^2$ yields exactly one graph for each $6 \leq n \leq 11$. However, it is not clear whether this pattern of uniqueness persists for $n \geq 12$. Additionally, in our visualization of polyhedral graphs, we added a vertex of degree d_1 to a graph with n vertices and drew a directed edge when the resulting structure matched a polyhedral graph with $n+1$ vertices. If it can be shown that this operation is only applicable in specific cases, then the possible shapes and counts of polyhedra corresponding to a given degree distribution may be determined theoretically.

Acknowledgments.

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