CCCG2025

The Number of Non-overlapping Unfoldings in Convex Polyhedra

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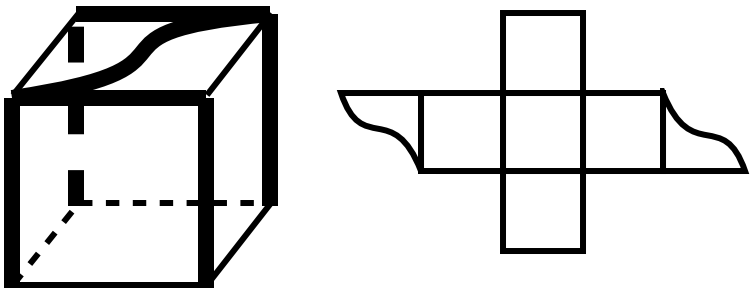
Unfoldings

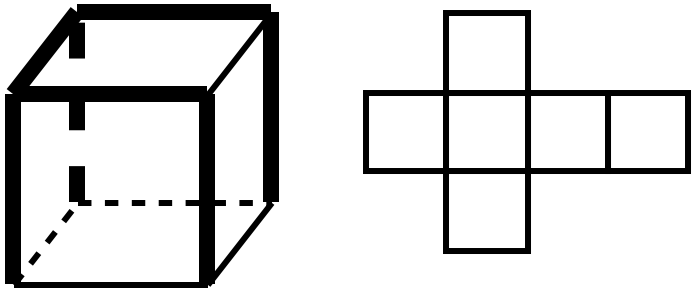
Definition 1 [E. D. Demaine and J. O'Rourke, 2007]

A general unfolding of a polyhedron is a flat polygon obtained by cutting along its edges and faces, then unfolding its faces into a plane. When the cuts are restricted to its edges, this is called an edge unfolding.

Cutting along the thick line on the left cube of each pair …



(a) General unfolding

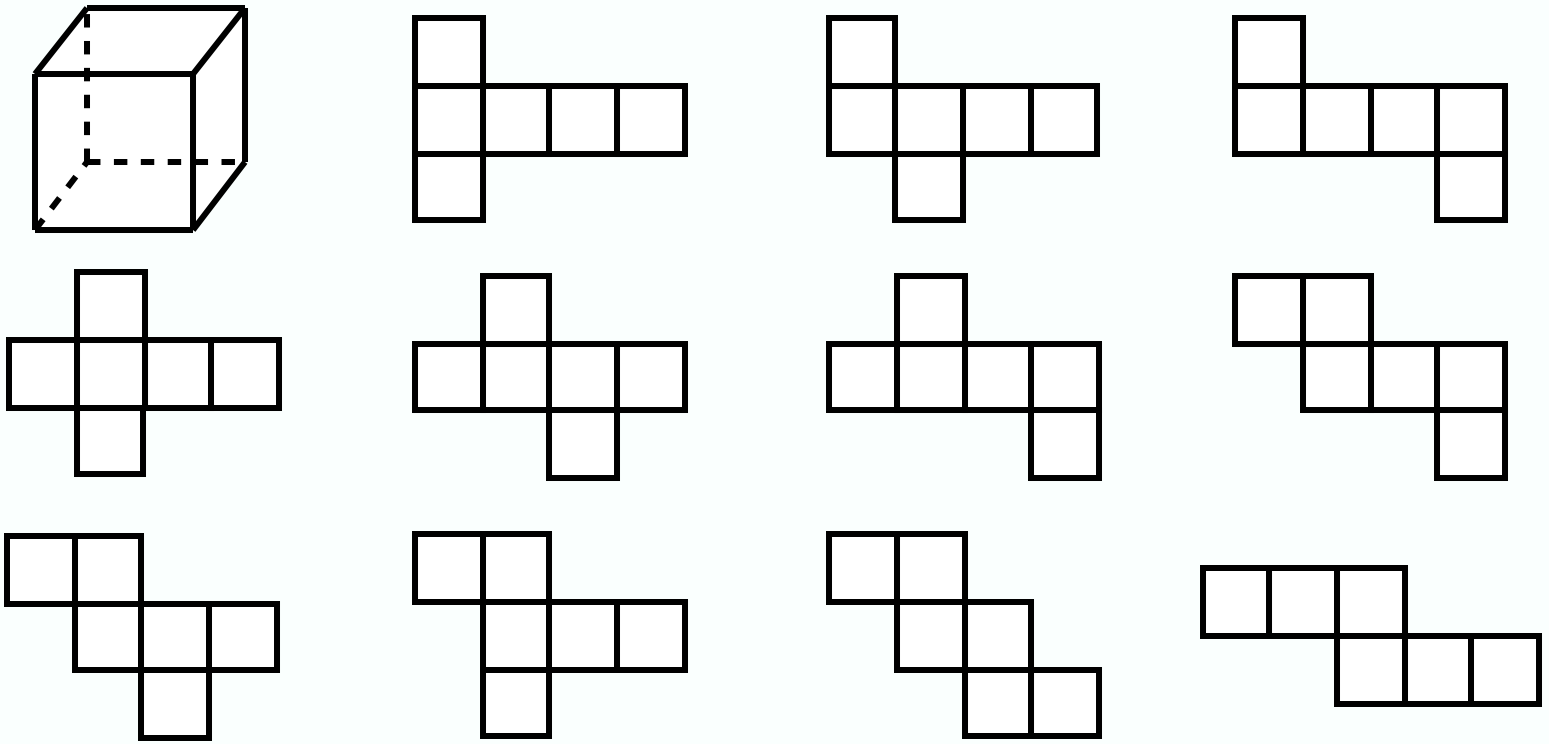


(b) Edge unfolding

#(Edge unfoldings)

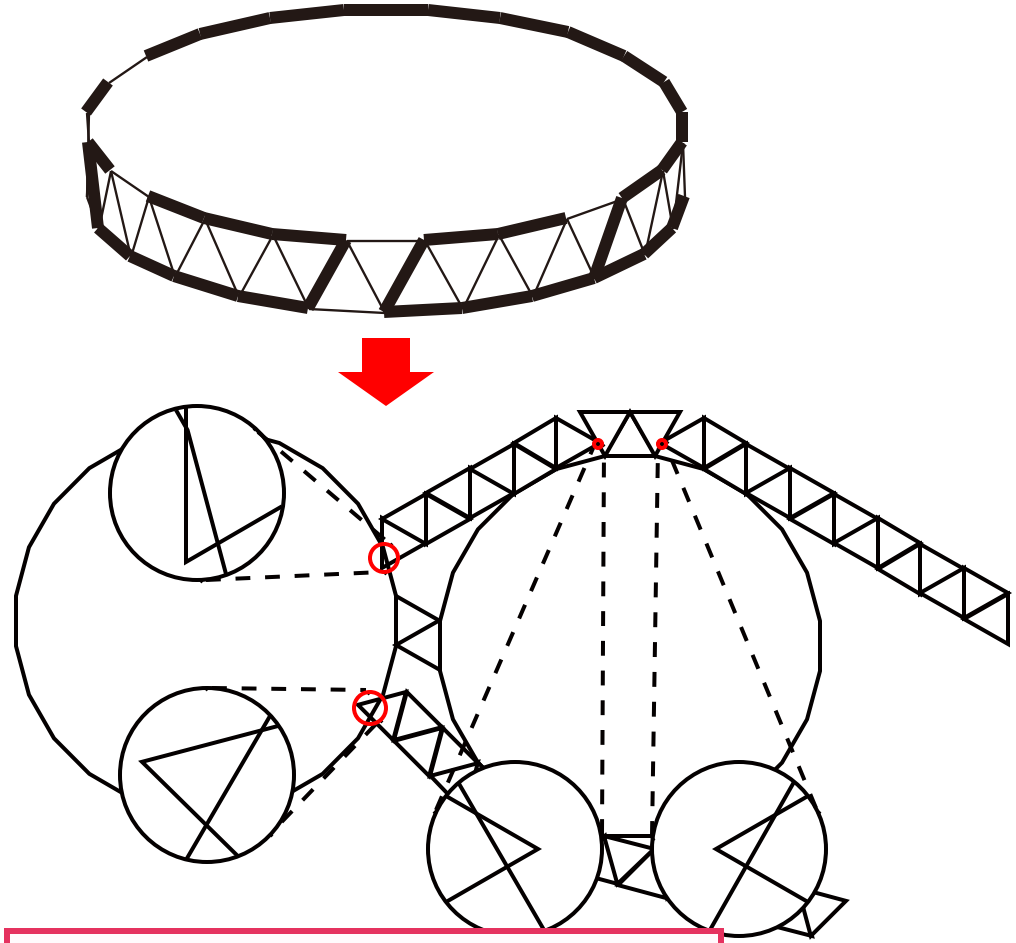
A polyhedron can get different edge unfoldings depending on which edges are cut.

[Ex.] A cube has 11 different edge unfoldings.



Overlapping / Non-overlapping

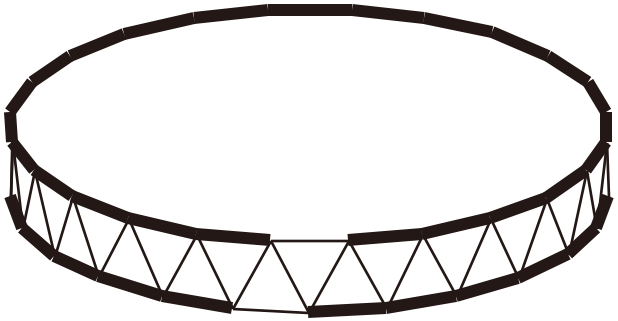
When the polyhedron is edge unfolded along the thick lines…

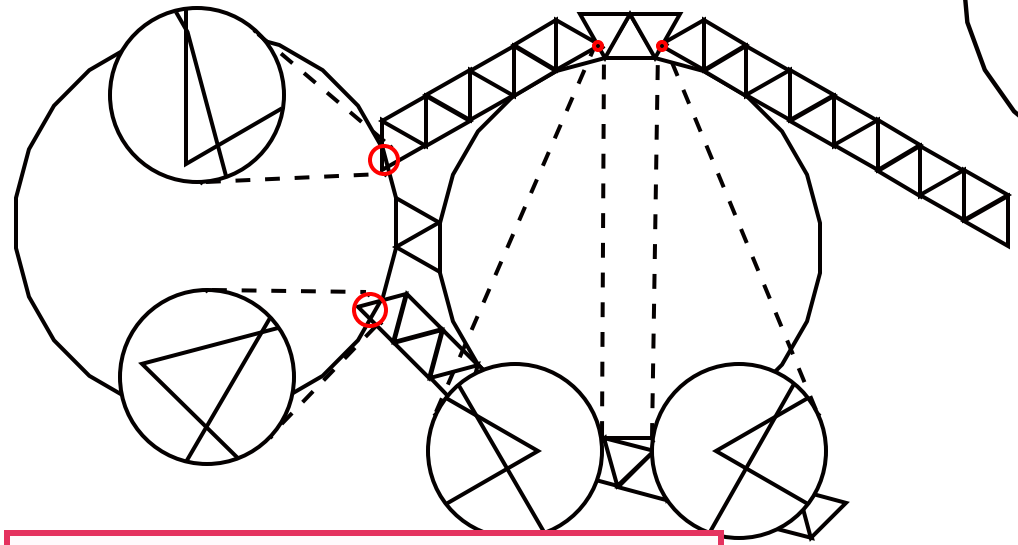


Overlapping unfolding

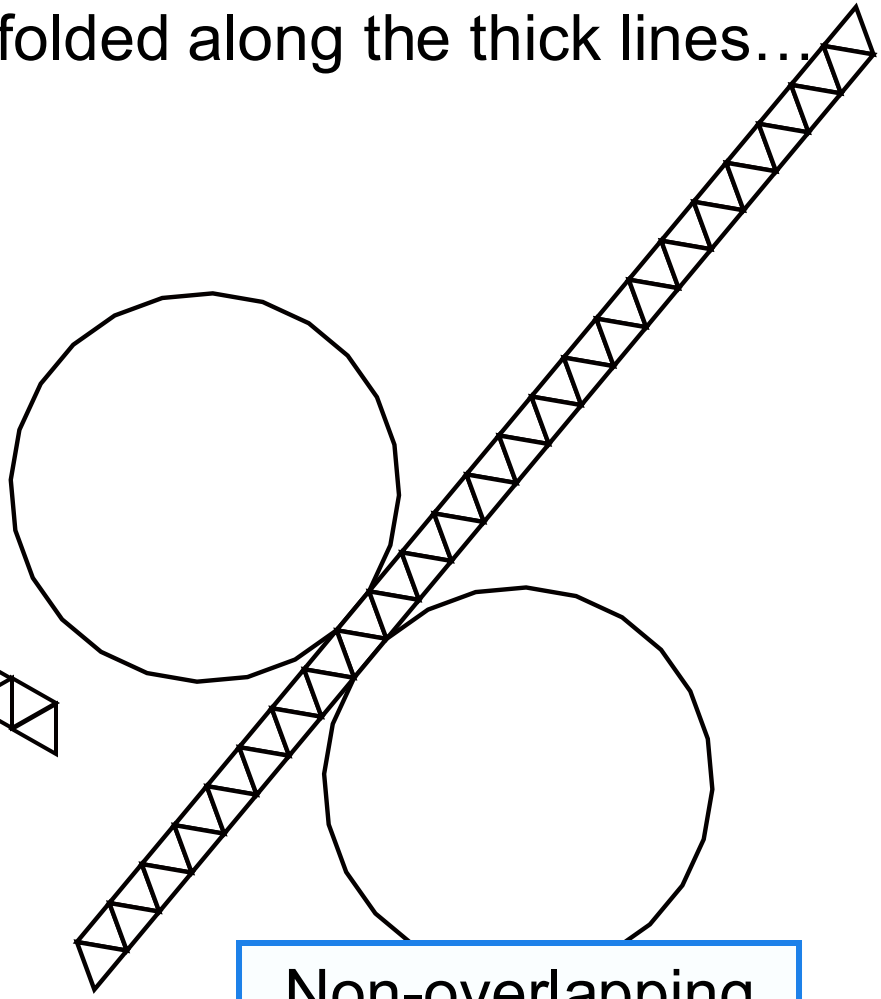
Overlapping / Non-overlapping

When the polyhedron is edge unfolded along the thick lines…



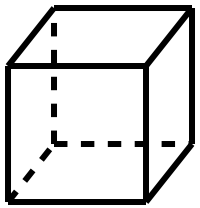
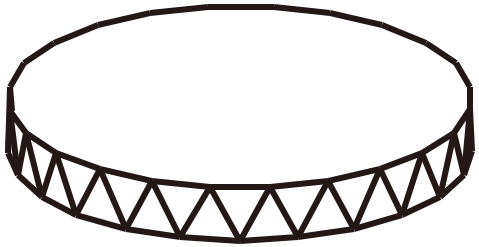


Overlapping unfolding



Non-overlapping unfolding

Overlapping / Non-overlapping

- Cube  ➡ No overlapping edge unfoldings
- Polyhedron on the previous page 

1,109,391,149,998,449,819,648
($\approx 1.1 \times 10^{21}$)

- ✔ Non-overlapping unfoldings
  - ➡ 162,941,846,136,285,049,392
- ✔ Overlapping unfoldings
  - ➡ 946,449,303,862,164,770,256

How many are there for each?

Main results of this study

We proposed an approach for counting the number of non-overlapping unfoldings.

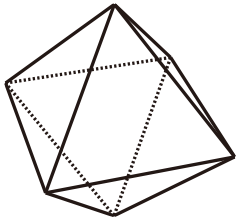
Target polyhedra

The existence of overlapping edge unfoldings is known.
[T. Horiyama and W. Shoji, 2011; T. Shiota and T. Saitoh, 2024; …]

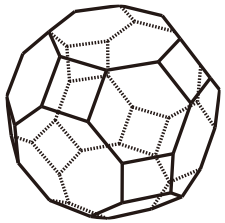
Defini

If all faces of a convex polyhedron are regular polygons, it is called a convex regular-faced polyhedron.

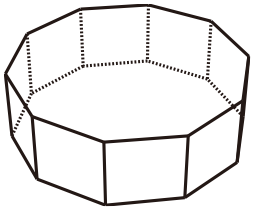
5 classes



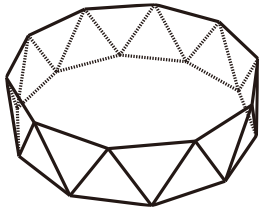
Platonic solids
(5 types)



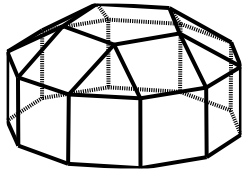
Archimedean solids
(13 types)



n-gonal Archimedean prisms
$(n \geq 3)$



m-gonal Archimedean antiprisms
$(m \geq 3)$



Johnson solids
(92 types)

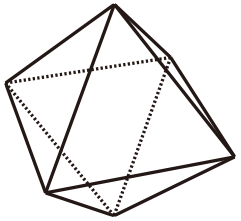
Target polyhedra

The existence of overlapping edge unfoldings is known.
[T. Horiyama and W. Shoji, 2011; T. Shiota and T. Saitoh, 2024; …]

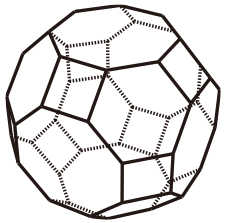
Defini

If all faces of a convex polyhedron are regular polygons, it is called a convex regular-faced polyhedron.

5 classes

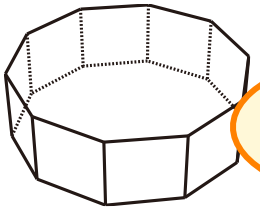


Platonic solids
(5 types / 0 types)



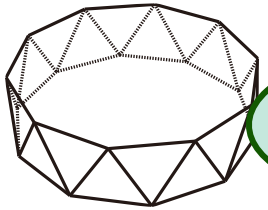
Archimedean solids
(7 types / 6 types)

3 types



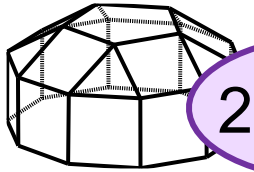
76 types

n-gonal Archimedean prisms
($3 \leq n \leq 23$ / $n \geq 24$)



68 types

m-gonal Archimedean antiprisms
($3 \leq m \leq 11$ / $m \geq 12$)

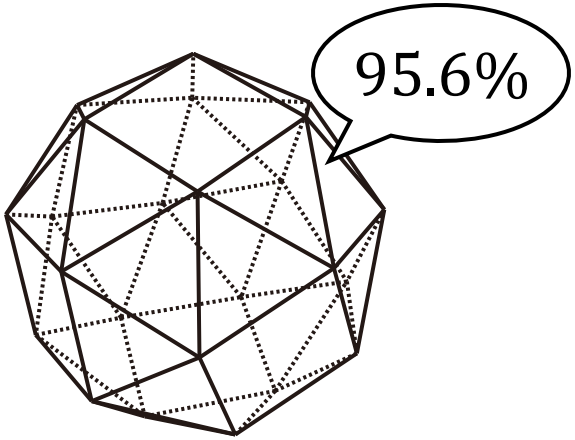


28 types

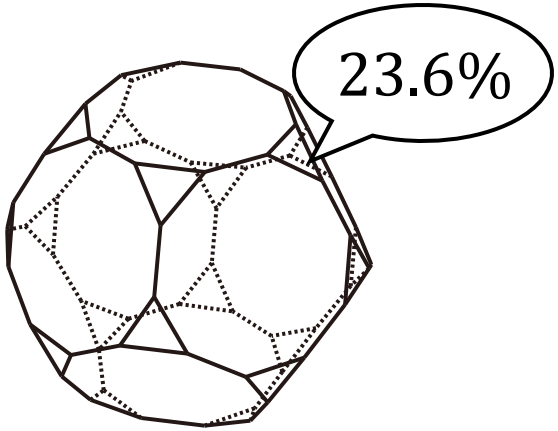
Johnson solids
(48 types / 44 types)

Results for Archimedean solids

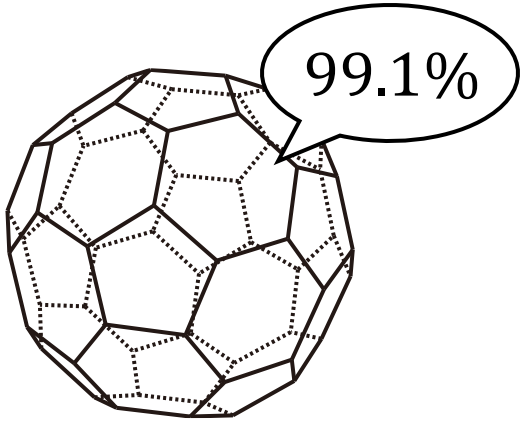
Archimedean solids	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
Snub cube	89,904,012,853,248	85,967,688,920,076
Truncated dodecahedron	4,982,259,375,000,000,000	1,173,681,002,295,455,040
Truncated icosahedron	375,291,866,372,898,816,000	371,723,160,733,469,233,260



Sunb cube



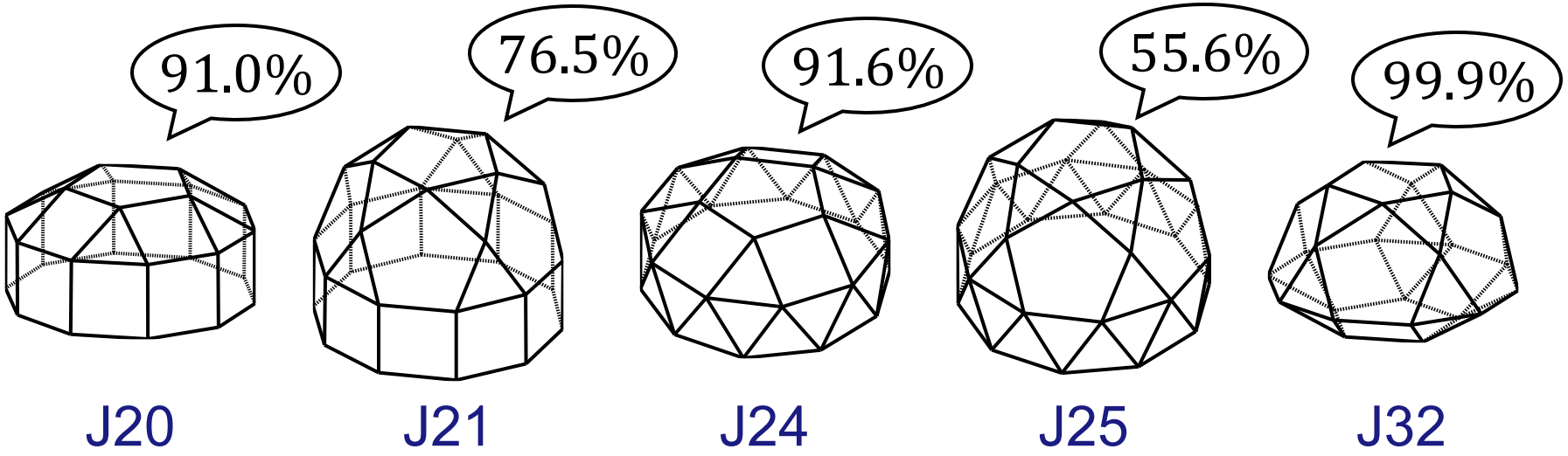
Truncated dodecahedron



Truncated icosahedron

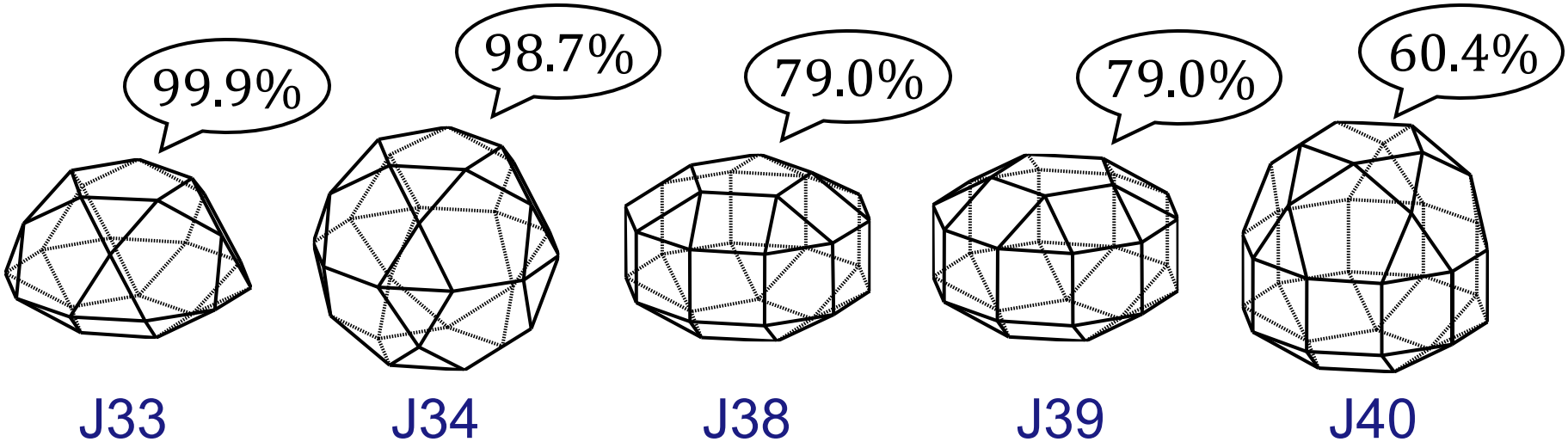
Results for Johnson solids (1)

Johnson solids	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
J20	29,821,320,745	27,158,087,415
J21	8,223,103,375,490	6,297,186,667,720
J24	5,996,600,870,820	5,492,624,228,190
J25	1,702,422,879,696,000	947,565,833,513,130
J32	699,537,024,120	699,433,603,320



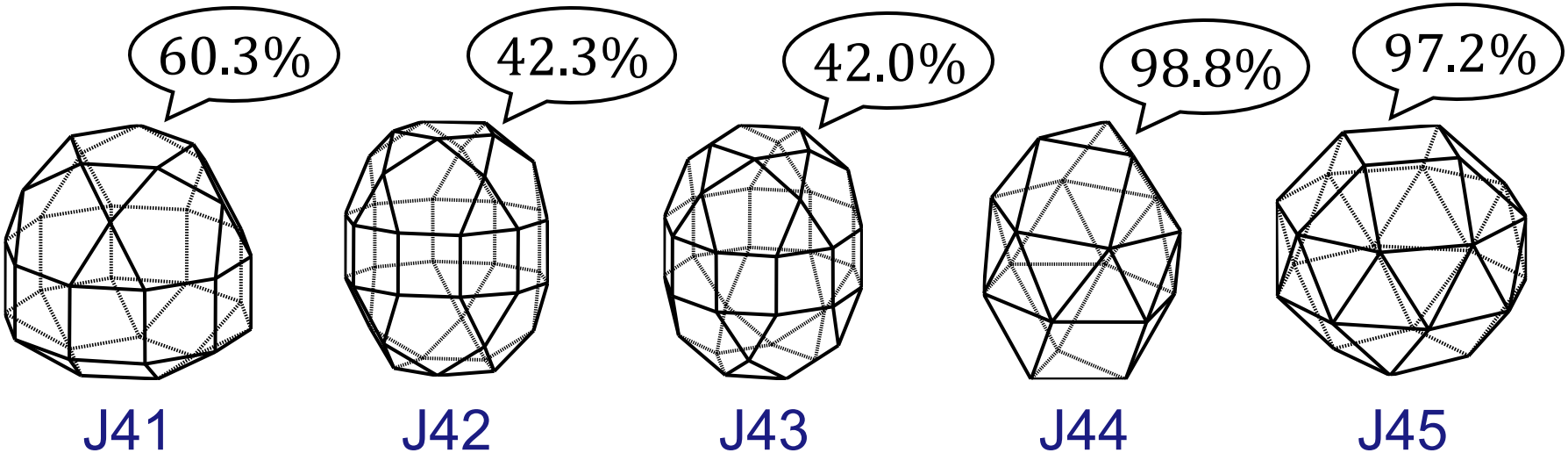
Results for Johnson solids (2)

Johnson solids	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
J33	745,208,449,920	745,105,029,120
J34	193,003,269,869,040	190,653,702,525,040
J38	270,745,016,304,350	214,085,775,357,270
J39	272,026,496,000,000	215,087,798,524,180
J40	75,378,202,163,880,700	45,541,858,035,543,690



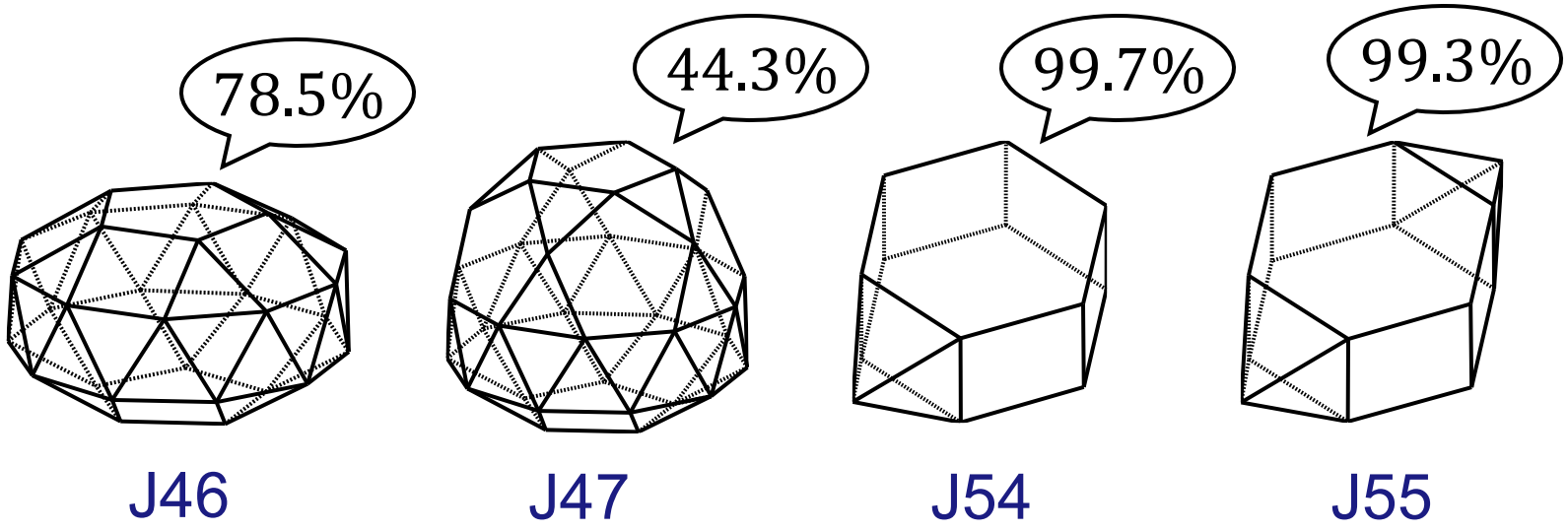
Results for Johnson solids (3)

Johnson solids	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
J41	75,804,411,381,317,500	45,774,968,967,924,850
J42	20,969,865,292,417,385,400	8,873,953,322,249,583,330
J43	21,115,350,368,078,435,000	8,884,490,741,507,534,860
J44	5,295,528,588	5,231,781,954
J45	13,769,880,349,680	13,386,219,088,644



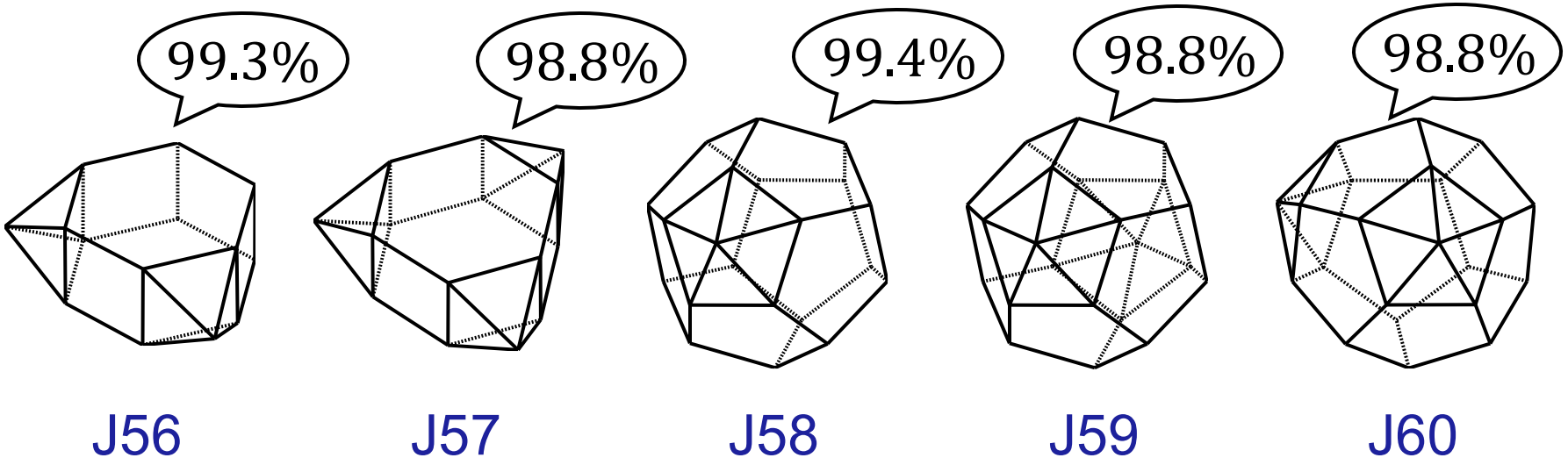
Results for Johnson solids (4)

Johnson solids	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
J46	32,543,644,773,848,180	25,553,553,814,333,235
J47	9,324,488,558,669,593,960	4,135,578,144,180,583,965
J54	75,973	75,749
J55	709,632	705,144



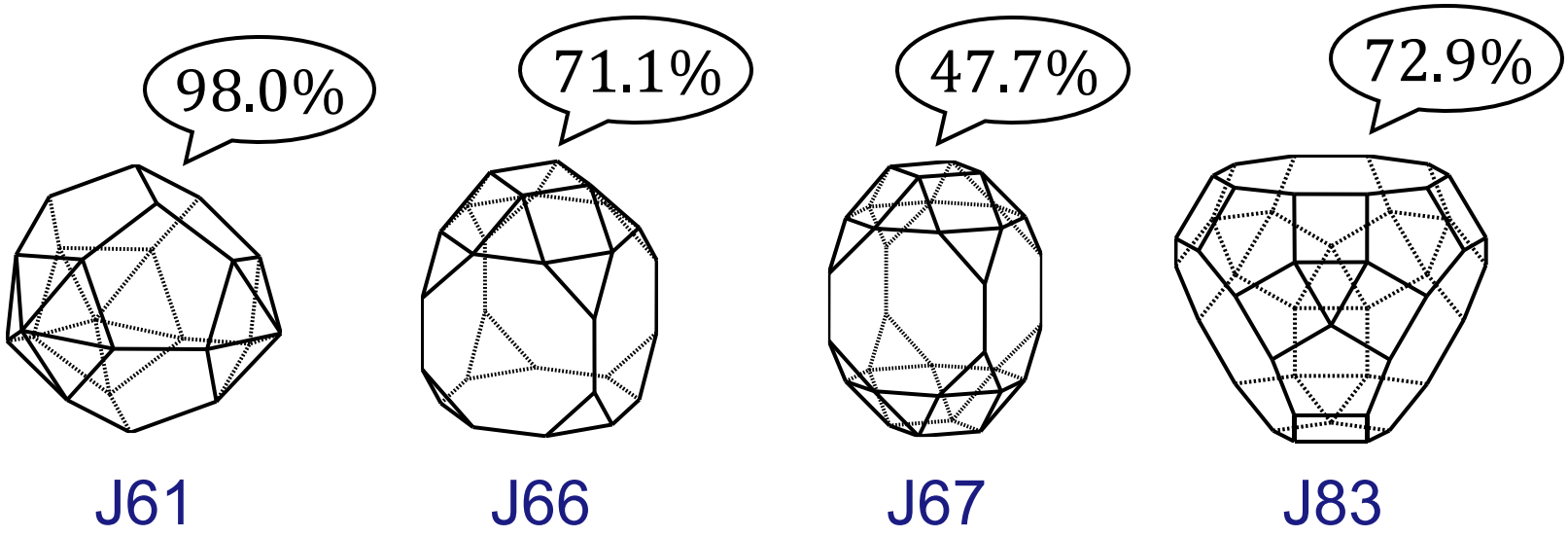
Results for Johnson solids (5)

Johnson solids	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
J56	707,232	702,520
J57	6,531,840	6,457,860
J58	92,724,962	92,219,782
J59	1,651,482,010	1,632,941,030
J60	1,641,317,568	1,621,738,522



Results for Johnson solids (6)

Johnson solids	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
J61	28,745,798,400	28,183,512,978
J66	54,921,311,280	39,055,563,000
J67	90,974,647,120,896	43,437,626,181,464
J83	197,148,908,795,401,104	143,844,293,105,396,598



Results for Archimedean prisms (1)

Prisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
24 prism	639,620,518,118,400	611,750,144,604,960
25 prism	2,486,558,615,814,025	2,378,211,063,753,525
26 prism	9,651,161,613,824,796	9,120,749,762,911,540
27 prism	37,403,957,244,654,675	35,348,297,730,550,335
28 prism	144,763,597,316,784,768	136,369,030,045,792,768
29 prism	559,560,282,425,278,229	377,763,966,359,384,333
30 prism	2,160,318,004,043,512,500	1,457,228,998,699,944,660
31 prism	8,331,163,769,982,715,231	5,619,734,416,791,278,823
32 prism	32,095,304,749,163,937,792	21,649,687,090,073,296,384
33 prism	123,524,473,883,545,449,825	83,322,661,319,000,341,161
34 prism	474,969,297,739,230,927,564	320,315,730,957,505,974,740
35 prism	1,824,745,126,233,358,110,635	1,224,788,877,353,311,603,655
36 prism	7,004,614,136,879,907,849,600	4,397,626,384,555,854,813,048
37 prism	26,867,730,730,869,118,775,917	16,841,247,868,506,593,664,113
38 prism	102,981,783,095,242,242,871,908	64,245,596,838,412,691,619,868
39 prism	394,447,279,575,099,709,694,775	245,972,761,433,859,004,882,155
40 prism	1,509,843,372,596,510,348,221,440	898,435,929,860,914,751,335,120
41 prism	5,775,682,482,451,356,835,464,761	3,436,774,701,162,733,316,551,373
42 prism	22,080,875,606,379,223,850,418,300	13,138,720,470,258,404,605,154,004

Results for Archimedean prisms (2)

Prisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
43 prism	84,369,019,868,106,350,841,057,283	49,292,119,107,345,418,821,464,335
44 prism	322,192,014,517,039,121,756,425,344	188,238,848,570,683,472,535,311,712
45 prism	1,229,765,080,878,981,092,880,253,125	718,483,241,070,056,103,676,962,705
46 prism	4,691,535,669,063,616,134,304,408,596	2,731,490,668,982,448,941,464,299,772
47 prism	17,889,680,992,955,476,025,801,057,807	10,224,003,439,425,442,252,695,897,017
48 prism	68,185,734,533,013,527,410,214,707,200	38,248,653,878,322,746,431,035,217,728
49 prism	259,774,138,662,539,598,798,853,632,529	142,720,047,356,681,793,679,714,971,049
50 prism	989,275,799,980,653,489,079,068,384,300	543,491,856,931,615,421,592,902,162,300
51 prism	3,765,868,099,190,667,877,509,098,288,475	2,068,859,848,320,328,618,349,489,286,879
52 prism	14,329,987,768,640,883,479,630,169,743,232	7,785,152,064,553,875,176,792,718,269,152
53 prism	54,508,708,624,877,734,355,711,282,194,973	29,613,250,778,119,135,586,129,657,023,707
54 prism	207,267,558,157,030,661,743,340,920,104,900	112,603,308,544,085,153,945,426,983,878,660
55 prism	787,857,744,058,382,475,503,456,540,986,855	425,875,823,447,530,794,507,732,415,080,195
56 prism	2,993,785,586,870,888,884,013,575,853,822,976	1,618,161,634,602,851,785,742,807,896,140,064
57 prism	11,372,477,058,547,594,072,637,405,171,464,425	6,144,514,101,474,823,054,337,696,869,965,123
58 prism	43,187,270,299,014,781,811,139,187,410,691,548	23,326,536,057,764,626,358,265,044,412,279,868
59 prism	163,956,002,289,170,289,778,245,356,488,769,459	88,555,357,584,032,953,873,552,819,154,790,741
60 prism	622,263,183,812,606,109,322,543,144,035,600,000	336,093,972,645,843,991,118,728,788,427,726,200
61 prism	2,361,023,114,629,354,318,988,404,829,601,017,461	1,153,365,956,554,130,834,065,584,588,770,576,537

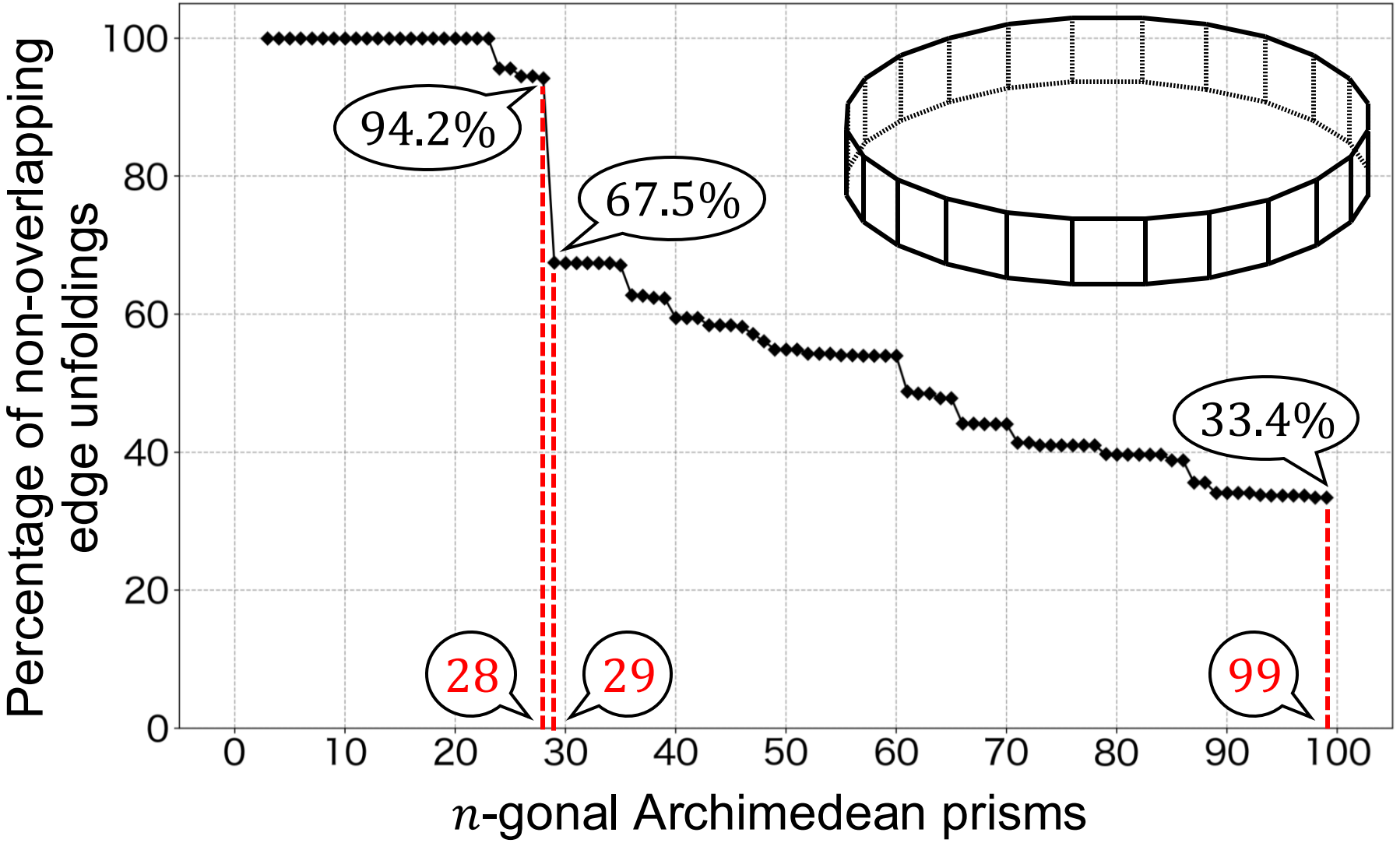
Results for Archimedean prisms (3)

Prisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
62 prism	8,955,908,356,422,272,120,516,285,708,666,803,572	4,348,799,175,168,145,604,642,783,894,710,257,972
63 prism	33,963,000,256,261,477,807,141,098,532,312,144,575	16,471,862,499,365,318,605,626,349,567,981,855,893
64 prism	128,763,573,367,713,152,730,420,340,995,267,231,744	61,639,673,042,788,410,952,524,482,230,015,782,656
65 prism	488,060,826,065,747,443,959,964,835,220,252,662,465	233,636,801,403,179,720,910,996,973,994,228,508,855
66 prism	1,849,490,381,600,812,352,868,765,046,397,041,481,100	817,502,763,787,586,935,738,546,005,495,007,277,736
67 prism	7,006,973,770,308,488,575,706,974,966,641,609,633,547	3,096,713,246,303,000,494,444,990,880,370,568,593,743
68 prism	26,540,686,328,811,552,652,967,327,238,752,884,476,288	11,708,731,916,039,177,215,090,282,288,662,448,445,864
69 prism	100,507,824,991,680,378,240,003,224,046,430,181,592,525	44,338,599,512,338,312,952,849,566,215,445,778,264,511
70 prism	380,536,545,795,702,174,419,400,936,760,625,367,754,020	167,871,805,778,112,195,880,603,660,146,778,531,028,040
71 prism	1,440,470,033,375,554,519,683,181,104,192,641,139,543,191	596,980,870,760,277,516,010,511,111,996,499,418,929,177
72 prism	5,451,624,356,286,428,491,183,290,436,982,561,065,036,800	2,259,342,205,174,960,128,925,954,191,858,205,229,812,848
73 prism	20,628,318,790,905,383,592,284,267,890,431,520,768,956,313	8,469,990,610,622,325,272,650,976,154,778,586,597,435,431
74 prism	78,040,535,635,296,089,880,020,963,154,546,570,729,579,324	32,025,918,206,891,338,090,282,229,066,878,434,112,166,032
75 prism	295,187,071,662,987,687,788,834,025,600,273,039,376,171,875	121,137,505,659,975,371,817,128,867,759,297,593,082,123,025
76 prism	1,116,341,857,839,528,524,717,385,720,706,815,646,963,560,576	458,116,563,229,230,551,652,298,491,316,191,672,953,377,976
77 prism	4,221,063,539,073,913,152,987,956,742,195,551,512,005,068,837	1,732,009,405,771,917,690,525,457,067,579,534,620,683,321,047
78 prism	15,957,810,909,148,397,191,421,362,206,489,368,194,976,255,700	6,547,759,140,478,958,708,285,931,868,337,379,111,699,013,260
79 prism	60,318,891,360,909,981,287,537,928,032,449,270,480,051,118,959	23,966,231,512,837,756,482,429,326,605,666,856,951,504,314,983
80 prism	227,962,700,977,360,477,553,905,172,759,643,132,779,913,338,880	90,516,297,016,221,842,120,588,697,662,660,340,385,338,212,160

Results for Archimedean prisms (4)

Prisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
81 prism	861,402,987,056,617,421,633,941,618,402,777,587,646,191,546,225	341,995,341,132,294,891,175,058,139,558,428,542,441,464,210,039
82 prism	3,254,488,598,838,582,230,210,121,899,601,749,106,776,524,965,612	1,292,101,188,372,407,798,075,371,911,903,984,534,028,933,754,436
83 prism	12,294,037,740,147,518,091,413,729,519,608,692,068,340,007,484,603	4,880,994,415,535,251,356,999,865,904,928,194,228,585,240,087,219
84 prism	46,434,768,337,561,243,483,045,269,738,500,630,193,132,229,526,400	18,435,535,468,052,500,560,475,959,629,177,515,836,675,305,835,264
85 prism	175,359,973,181,486,662,638,962,060,133,459,799,581,503,709,666,685	68,136,242,011,705,283,282,291,593,247,449,255,410,135,074,971,535
86 prism	662,151,768,698,132,480,917,981,340,031,934,840,615,593,080,769,156	257,261,176,564,360,166,567,970,609,875,291,471,141,053,287,325,288
87 prism	2,499,918,741,278,642,349,615,482,066,241,598,681,934,395,700,879,175	890,564,328,098,014,416,825,940,792,706,208,682,651,794,408,576,921
88 prism	9,437,063,110,777,086,198,028,843,620,687,140,090,853,387,600,995,328	3,361,807,045,468,520,205,795,018,776,129,264,881,669,261,524,635,520
89 prism	35,619,821,719,604,700,475,856,960,433,270,363,377,351,219,128,623,689	12,147,147,750,557,166,009,277,266,446,164,341,729,697,940,171,357,263
90 prism	134,428,635,924,381,058,558,342,373,483,695,239,998,308,348,737,337,500	45,835,667,167,756,167,327,808,684,200,497,934,083,305,406,003,263,080
91 prism	507,268,882,587,101,907,135,928,966,969,950,901,239,037,550,690,556,691	172,961,693,284,074,318,802,898,801,630,921,272,808,376,171,618,605,053
92 prism	1,913,957,124,704,016,720,646,095,852,898,568,656,870,706,493,680,372,352	652,587,801,648,072,657,385,242,064,252,120,087,145,845,127,550,204,344
93 prism	7,220,626,376,739,743,204,712,927,445,165,724,589,272,091,607,783,034,325	2,440,735,263,004,787,039,318,901,058,435,443,959,679,578,395,810,067,499
94 prism	27,237,505,194,018,078,864,602,783,571,427,472,860,418,008,032,804,687,924	9,200,983,829,108,871,720,824,454,037,812,685,220,751,863,003,740,596,660
95 prism	102,733,154,885,874,285,090,022,412,414,655,338,630,173,029,721,095,307,295	34,703,835,672,760,725,659,254,115,573,775,525,383,657,603,505,506,462,795
96 prism	387,441,199,483,882,790,074,386,518,739,975,425,611,166,119,546,861,977,600	130,879,809,160,332,709,730,909,060,139,747,285,756,827,067,222,934,653,120
97 prism	1,461,012,223,100,730,076,686,798,067,582,261,367,840,286,394,049,834,758,177	493,533,958,003,824,346,986,768,858,213,124,355,417,772,366,581,533,141,923
98 prism	5,508,783,927,988,926,594,011,003,719,084,348,517,902,267,230,291,120,238,668	1,842,484,668,472,842,688,235,271,932,872,735,062,103,648,776,666,005,382,528
99 prism	20,768,847,849,083,459,407,230,734,273,699,231,588,352,138,676,085,736,775,275	6,946,403,315,106,848,273,582,090,103,916,680,058,367,861,„235,272,392,879,353

Results for Archimedean prisms



Results for Archimedean antiprisms (1)

Antiprisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
12 antiprism	51,599,794,176	49,743,531,024
13 antiprism	383,142,771,674	369,359,503,344
14 antiprism	2,828,107,288,188	2,726,368,290,352
15 antiprism	20,768,716,848,000	20,021,578,135,380
16 antiprism	151,840,963,183,392	146,378,600,602,880
17 antiprism	1,105,779,284,582,146	989,008,190,008,480
18 antiprism	8,024,954,790,380,544	1,517,682,139,108,200
19 antiprism	58,059,628,319,357,318	10,550,126,657,845,736
20 antiprism	418,891,171,182,561,000	72,542,787,706,846,320
21 antiprism	3,014,678,940,049,375,872	500,034,989,831,068,818
22 antiprism	21,646,865,272,061,272,716	3,449,844,625,120,946,448
23 antiprism	155,113,904,634,576,144,814	23,752,014,262,731,255,118
24 antiprism	1,109,391,149,998,449,819,648	162,941,846,136,285,049,392
25 antiprism	7,920,708,398,483,722,531,250	1,117,782,108,867,439,830,950
26 antiprism	56,460,916,728,463,179,389,652	7,655,723,643,342,875,568,936
27 antiprism	401,873,068,071,158,383,691,136	40,561,091,359,603,932,081,708
28 antiprism	2,856,496,726,273,368,888,420,984	264,478,642,931,290,919,674,648

Results for Archimedean antiprisms (2)

Antiprisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
29 antiprism	20,277,959,821,998,087,658,569,178	1,739,241,575,214,473,110,359,470
30 antiprism	143,779,866,504,299,168,102,784,000	11,718,403,001,480,040,992,138,460
31 antiprism	1,018,331,261,238,041,888,906,149,982	79,227,778,390,260,949,681,648,022
32 antiprism	7,204,899,406,395,028,729,775,662,656	535,088,808,567,353,165,252,109,504
33 antiprism	50,926,337,537,628,456,148,426,034,304	3,609,765,503,967,786,066,361,340,190
34 antiprism	359,631,713,591,480,208,135,988,999,908	23,021,698,690,971,475,731,990,378,832
35 antiprism	2,537,451,036,289,964,010,662,071,375,750	155,062,662,340,102,785,066,461,404,560
36 antiprism	17,888,860,941,014,408,891,681,749,082,112	1,043,552,488,055,809,272,918,087,425,832
37 antiprism	126,017,967,976,156,654,397,534,266,950,026	6,992,936,456,851,489,392,078,163,320,978
38 antiprism	887,084,326,468,926,324,030,843,544,372,524	44,438,923,337,128,563,356,651,187,920,012
39 antiprism	6,240,170,805,918,890,922,630,444,422,537,088	297,478,382,005,821,691,364,579,412,449,706
40 antiprism	43,867,453,323,674,409,143,926,999,140,738,000	1,983,085,398,586,205,226,911,001,715,311,520
41 antiprism	308,188,798,032,167,102,842,859,597,775,205,042	13,033,685,199,501,916,122,251,960,215,169,912
42 antiprism	2,163,878,359,899,340,120,052,552,791,046,378,496	83,997,191,224,353,764,482,646,914,913,598,624
43 antiprism	15,184,572,514,675,762,272,842,247,131,661,635,894	557,611,275,066,805,327,907,702,044,873,987,094
44 antiprism	106,496,994,569,720,990,727,767,374,869,609,730,968	3,729,635,546,799,294,784,771,659,621,890,472,664
45 antiprism	746,530,833,968,188,588,851,681,523,936,666,896,000	24,431,729,185,673,950,396,752,450,635,669,388,990

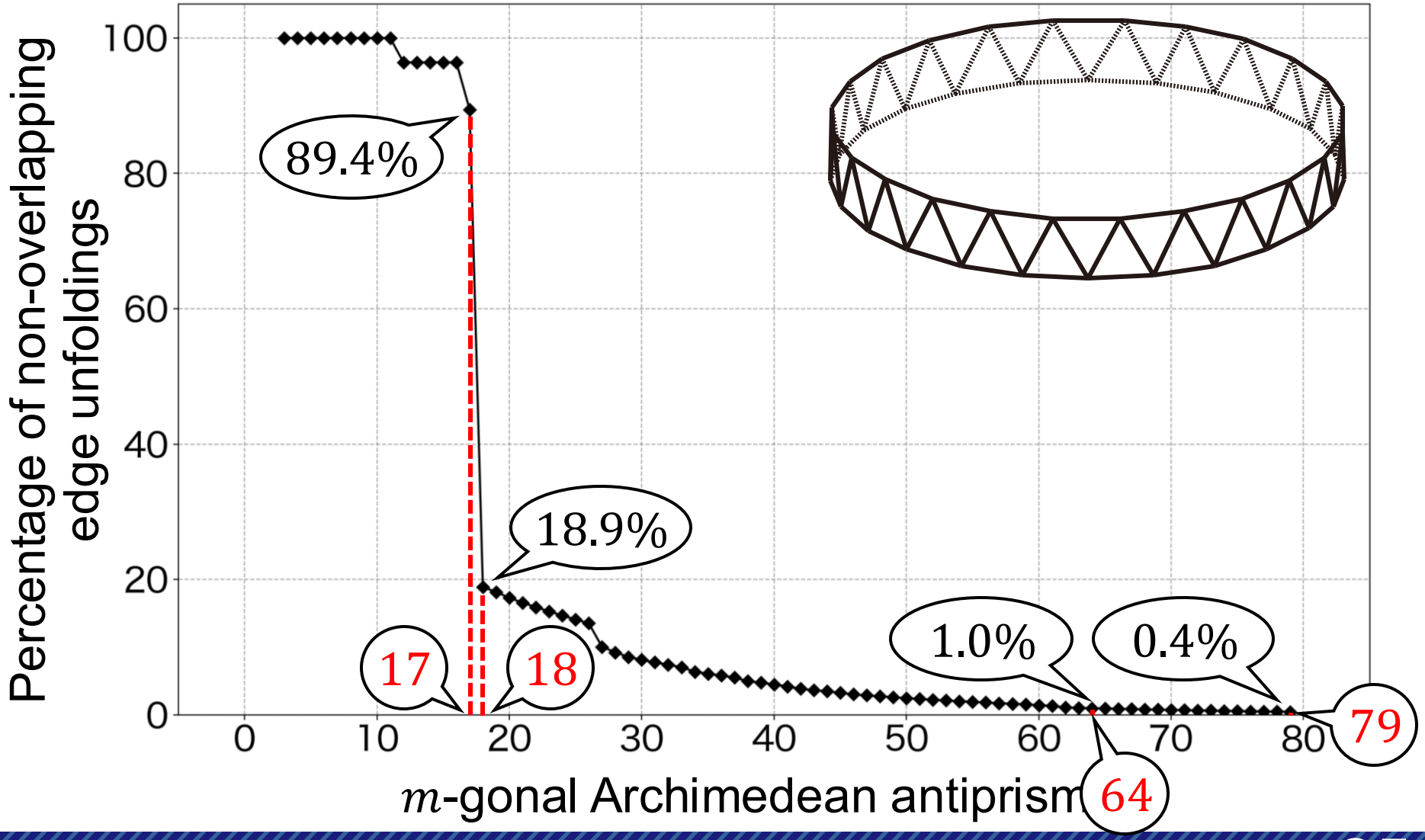
Results for Archimedean antiprisms (3)

Antiprisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
46 antiprism	5,230,505,089,344,431,048,507,144,123,815,456,787,772	158,880,058,222,038,902,280,026,924,999,326,487,380
47 antiprism	36,629,772,069,905,834,755,580,161,013,689,257,929,566	1,060,539,783,181,405,072,337,781,040,938,759,619,342
48 antiprism	256,405,984,103,100,622,357,453,677,837,305,041,649,664	7,079,805,121,439,383,694,342,635,647,608,115,253,632
49 antiprism	1,794,045,942,295,064,986,560,011,614,233,598,156,819,298	47,081,894,856,079,882,568,065,465,270,204,379,835,426
50 antiprism	12,547,524,306,762,115,327,139,640,139,635,651,225,562,500	313,979,052,982,335,359,074,797,901,078,961,675,923,700
51 antiprism	87,722,051,242,994,803,143,643,140,957,694,192,485,255,552	2,093,220,753,862,771,345,181,080,442,901,094,417,625,334
52 antiprism	613,045,214,965,087,171,516,365,035,207,733,551,443,709,736	13,847,372,368,702,386,134,230,159,841,426,631,160,249,520
53 antiprism	4,282,679,690,470,859,990,496,705,254,406,531,422,464,957,834	92,254,873,128,059,280,848,212,456,691,530,042,110,511,660
54 antiprism	29,907,770,896,467,759,248,303,121,099,365,111,834,448,227,328	612,564,147,448,713,555,111,946,603,958,121,691,610,818,284
55 antiprism	208,787,039,294,802,995,558,997,194,768,329,038,664,289,012,750	4,066,189,399,259,848,921,000,213,749,369,244,891,377,708,200
56 antiprism	1,457,066,704,859,013,168,857,939,059,215,469,544,106,252,283,632	26,882,734,603,660,531,639,929,002,113,295,431,420,663,838,960
57 antiprism	10,165,220,976,851,309,359,988,036,885,968,497,679,868,602,544,256	177,996,738,793,784,313,663,675,391,748,841,946,567,960,370,322
58 antiprism	70,895,802,507,339,433,606,655,281,989,227,262,198,207,167,104,404	1,180,051,219,817,443,505,617,832,090,129,322,320,242,838,961,804
59 antiprism	494,305,112,112,066,674,502,236,216,299,036,533,680,424,852,661,558	7,834,254,664,590,050,866,382,827,703,061,557,850,624,078,276,054
60 antiprism	3,445,441,668,665,681,646,962,862,224,080,264,597,391,436,598,272,000	50,403,377,478,963,170,902,563,467,427,787,499,348,682,069,083,400
61 antiprism	24,008,998,657,730,043,418,999,210,016,836,054,318,873,476,626,984,762	326,358,125,570,791,131,913,614,291,766,990,868,139,803,779,993,658
62 antiprism	167,257,831,873,332,437,880,307,618,553,808,550,206,593,147,036,070,876	1,854,939,808,105,063,172,332,156,437,560,469,843,454,970,270,135,872

Results for Archimedean antiprisms (4)

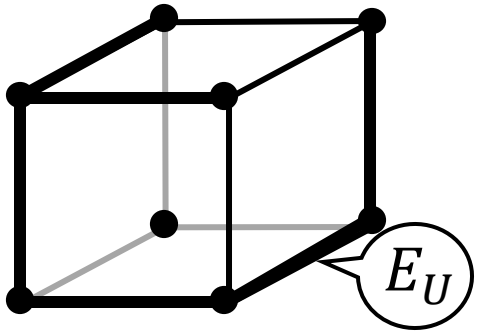
Antiprisms	#(Edge unfoldings) [T. Horiyama and W. Shoji, 2013]	#(Non-overlapping edge unfoldings)
63 antiprism	1,164,892,592,931,629,392,338,324,783,815,505,600,459,556,438,051,914,624	12,307,784,791,893,071,962,543,568,932,644,431,739,514,519,035,956,010
64 antiprism	8,111,027,415,042,412,087,059,884,505,184,466,841,764,376,870,066,703,488	81,711,622,221,644,705,528,593,850,868,389,602,010,305,415,118,773,632
65 antiprism	56,462,462,218,649,594,296,489,126,547,728,233,809,837,328,422,164,209,250	542,324,566,561,175,718,389,472,139,379,279,869,381,405,090,020,827,940
66 antiprism	392,953,311,363,100,782,765,649,901,258,650,734,263,966,574,522,587,784,192	3,596,222,163,944,667,874,762,237,572,824,582,243,007,524,492,920,016,364
67 antiprism	2,734,150,277,149,943,789,424,653,221,975,737,385,578,070,013,528,736,239,846	23,843,879,380,617,148,795,466,186,545,241,390,034,312,194,471,190,721,410
68 antiprism	19,019,848,444,227,125,038,960,788,030,632,146,757,126,643,109,790,562,588,744	158,047,277,673,032,158,023,557,741,664,008,526,923,236,605,161,466,417,144
69 antiprism	132,281,097,981,397,378,649,216,327,599,160,829,981,177,127,561,996,392,778,368	1,047,172,473,392,558,836,251,756,065,865,616,682,360,236,358,525,772,236,994
70 antiprism	919,808,251,652,716,036,371,207,892,032,413,807,786,850,000,883,510,526,443,500	6,,940,458,038,739,497,932,445,601,709,547,911,589,835,564,772,106,776,166,220
71 antiprism	6,394,523,254,028,788,004,842,904,160,103,465,297,190,061,440,230,870,478,513,422	45,780,684,032,708,802,155,441,190,837,474,021,683,024,682,589,684,818,605,684
72 antiprism	44,446,020,245,409,850,388,919,123,546,153,029,436,510,863,260,931,308,786,876,416	303,133,167,415,275,991,480,997,730,729,891,095,322,066,594,918,858,437,619,312
73 antiprism	308,868,631,905,426,344,328,427,311,641,720,314,433,493,675,086,071,079,589,514,514	1,993,862,494,867,116,801,892,228,563,979,771,114,995,027,333,450,860,247,871,268
74 antiprism	2,146,017,331,464,816,766,794,536,512,479,696,001,159,731,435,059,847,042,295,248,948	13,209,556,361,833,768,036,797,427,669,629,105,268,373,114,777,374,196,971,004,696
75 antiprism	14,907,792,173,512,400,291,813,743,106,474,046,926,942,024,685,661,040,985,606,000,000	87,479,366,310,664,176,876,248,551,641,616,300,775,155,031,501,800,307,738,254,650
76 antiprism	103,541,921,350,898,507,949,080,104,593,610,092,290,674,442,999,146,438,446,892,628,952	575,728,071,259,409,983,216,254,627,908,333,000,969,928,942,023,795,016,355,188,312
77 antiprism	719,024,872,072,048,704,795,415,649,743,438,922,497,201,689,435,088,835,503,134,774,586	3,803,094,888,814,534,194,244,711,198,507,594,930,371,276,487,989,070,769,505,333,618
78 antiprism	4,992,273,293,210,566,749,051,331,629,112,187,169,112,455,673,298,970,063,008,551,517,184	25,178,232,163,989,764,173,902,561,452,762,553,210,080,327,618,722,145,846,907,592,560
79 antiprism	34,656,236,736,012,517,253,147,752,391,859,302,565,359,593,442,298,439,368,730,738,540,478	166,652,766,549,703,045,550,278,222,766,074,887,734,370,837,666,827,789,428,184,155,666

Results for Archimedean antiprisms

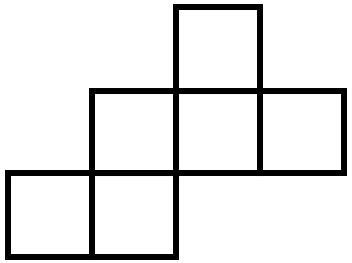


#(Edge unfoldings) include overlaps

Let V be the set of vertices and E the set of edges. Then, Q can be seen as a graph $G_Q = (V, E)$.



Polyhedron Q



Edge unfolding U

Theorem 3 [E. D. Demaine et al., 2007]

Let $E_U (\subset E)$ be the set of edges cut to obtain an unfolding U. Then, E_U corresponds to a spanning tree in G_Q.

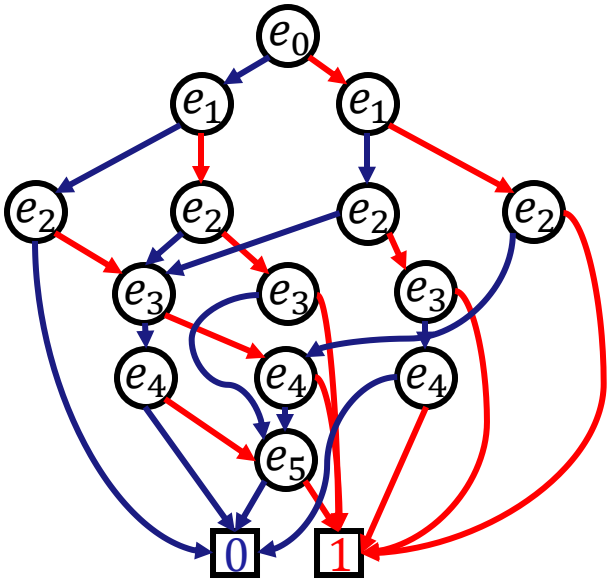
Key point:

If we want to get the number of edge unfoldings, we can count the number of spanning trees in G_Q instead.

#(Edge unfoldings) include overlaps

- #(Spanning trees in G) can be counted using ZDDs.

[J. Kawahara et al., 2017]



Zero-suppressed Decision Diagram (ZDD) [S. Minato, 1993]

A data structure that compactly represents a family of sets as a directed acyclic graph (DAG).

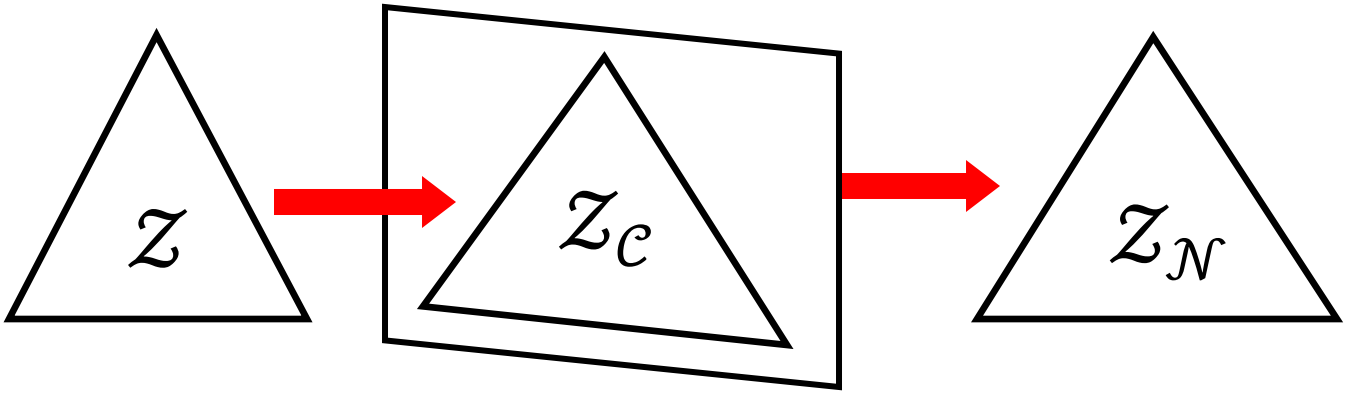
- We can use various operations over ZDDs.

$$\mathcal{Z}_{\mathcal{A}} \diamondsuit \mathcal{Z}_{\mathcal{B}} = \mathcal{Z}_{\mathcal{N}} \qquad \diamondsuit = \{\cup, \cap, \setminus, \ldots\}$$

Operations over ZDDs

Subsetting technique [H. Iwashita and S. Minato, 2013]

Constructs a new ZDD $\mathcal{Z}_{\mathcal{N}}$ by extracting a family of sets from an existing ZDD $\mathcal{Z}$ that satisfies a constraint $\mathcal{C}$.

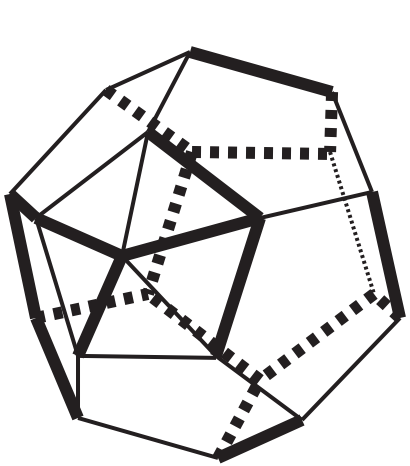


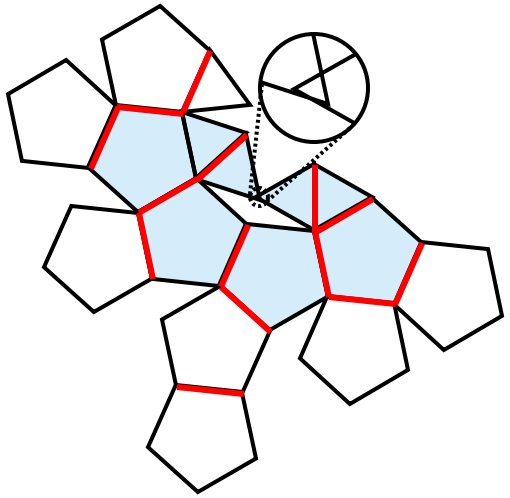
Key point:

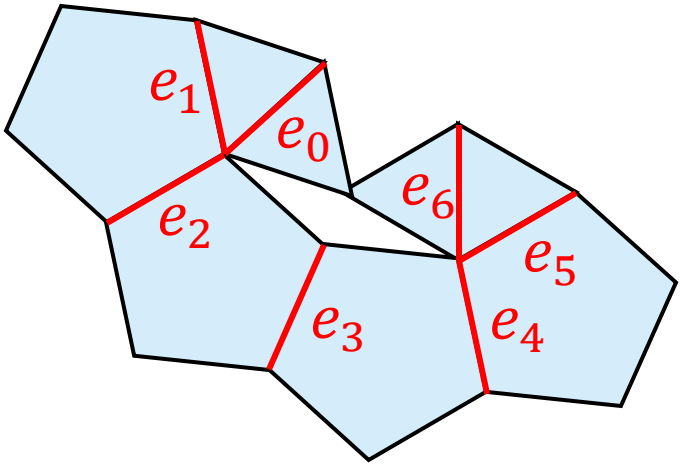
Using the subsetting technique, we removed overlapping edge unfoldings and counted #(non-overlapping edge unfoldings).

#(Non-overlapping edge unfoldings)

[Ex.] Overlapping edge unfolding U_1
[T. Shiota and T. Saitoh, 2024]







Minimal overlapping partial unfolding M_1

MOPU

$NC[U]$: Set of edges that are not cut in unfolding U

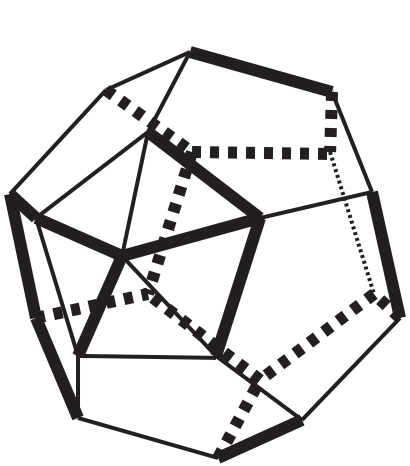
$NC[U_1] = \{e_0, e_1, \dots, e_6, e_7, \dots, e_{14}\}$ $NC[M_1] = \{e_0, e_1, \dots, e_6\}$

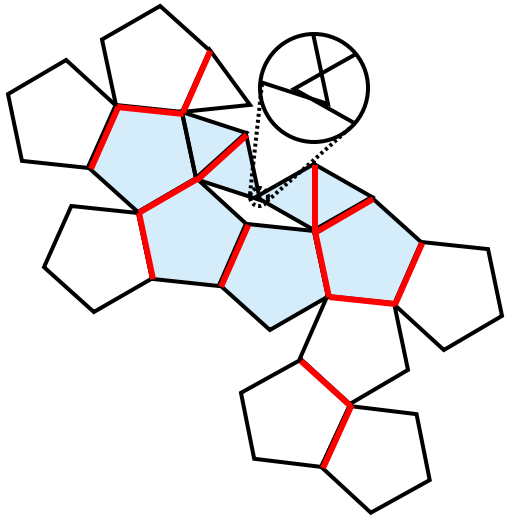
Key point:

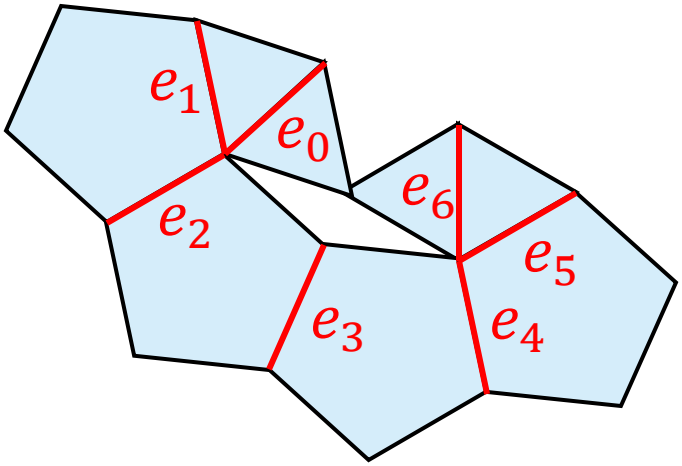
If $NC[M_1] \subseteq NC[U]$, the edge unfoldings always overlaps.

#(Non-overlapping edge unfoldings)

[Ex.] Overlapping edge unfolding U_1
[T. Shiota and T. Saitoh, 2024]







Minimal overlapping partial unfolding M_1

MOPU

$NC[U]$: Set of edges that are not cut in unfolding U

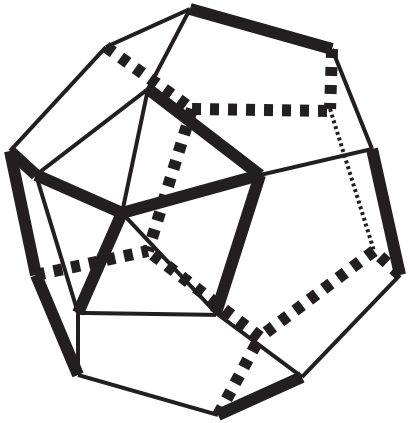
$NC[U_1] = \{e_0, e_1, \dots, e_6, e_7, \dots, x\}$ $\quad NC[M_1] = \{e_0, e_1, \dots, e_6\}$

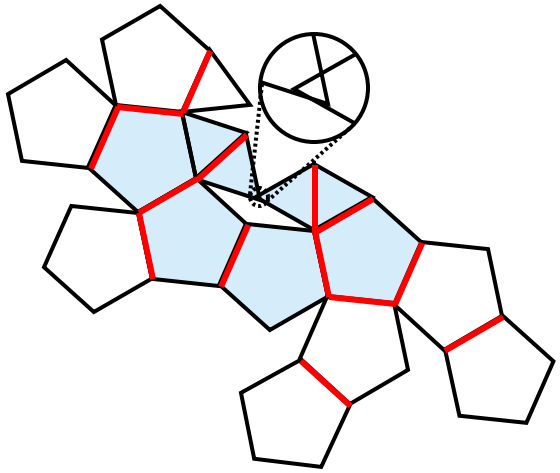
Key point:

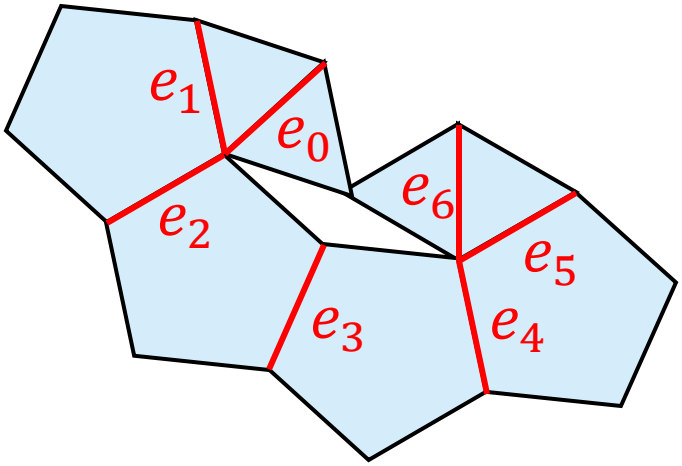
If $NC[M_1] \subseteq NC[U]$, the edge unfoldings always overlaps.

#(Non-overlapping edge unfoldings)

[Ex.] Overlapping edge unfolding U_1
[T. Shiota and T. Saitoh, 2024]







Minimal overlapping partial unfolding M_1

MOPU

$NC[U]$: Set of edges that are not cut in unfolding U

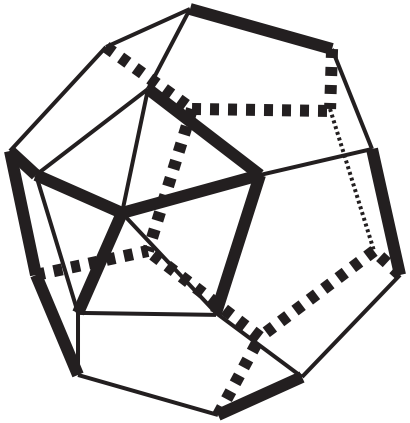
$NC[U_1] = \{e_0, e_1, \dots, e_6, a, \dots, x\}$ $\quad$ $NC[M_1] = \{e_0, e_1, \dots, e_6\}$

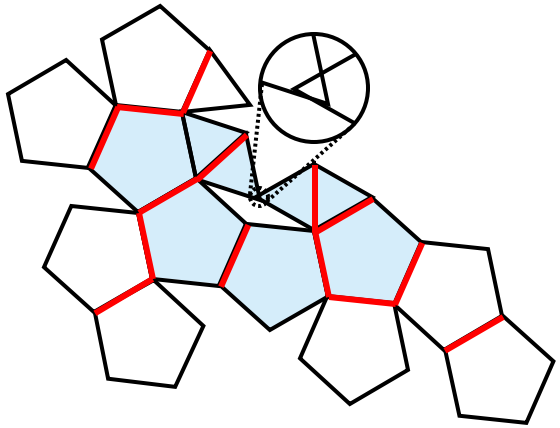
Key point:

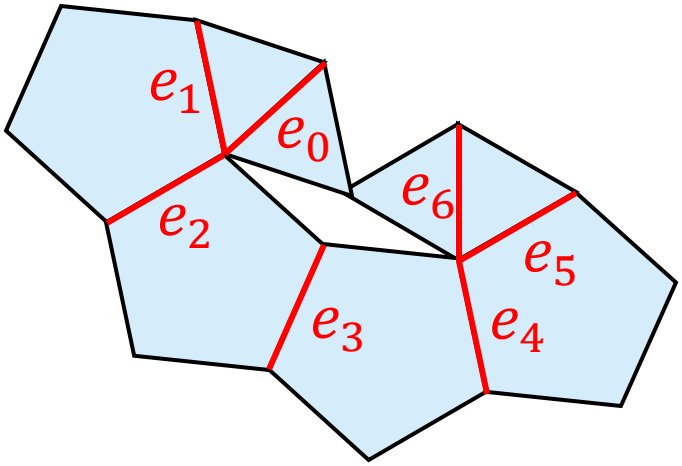
If $NC[M_1] \subseteq NC[U]$, the edge unfoldings always overlaps.

#(Non-overlapping edge unfoldings)

[Ex.] Overlapping edge unfolding U_1
[T. Shiota and T. Saitoh, 2024]







Minimal overlapping partial unfolding M_1

MOPU

$NC[U]$: Set of edges that are not cut in unfolding U

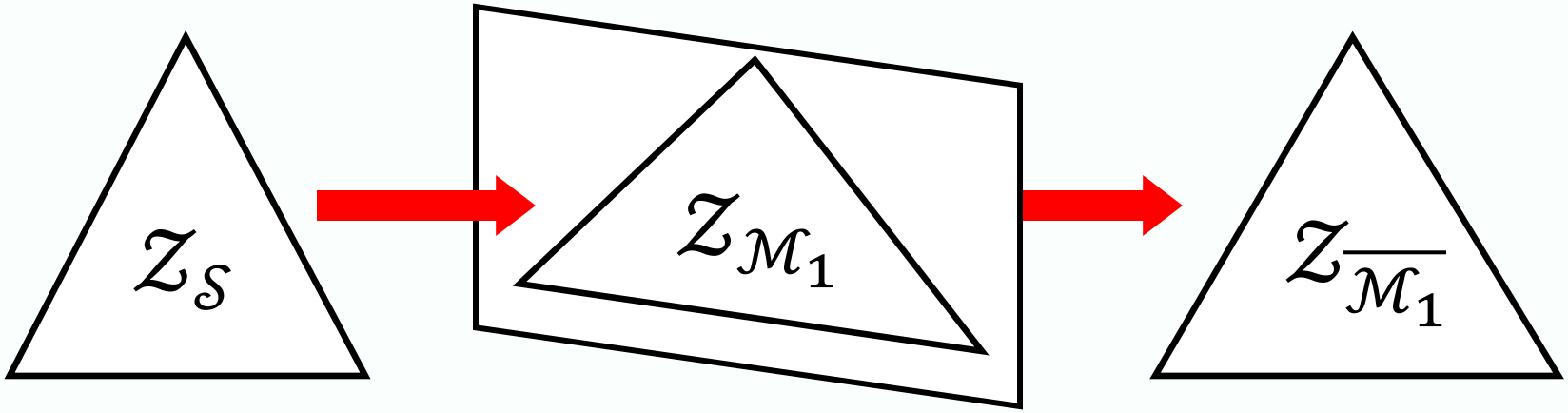
$NC[U_1] = \{e_0, e_1, \dots, e_6, a, \dots, y\}$ $NC[M_1] = \{e_0, e_1, \dots, e_6\}$

Key point:

If $NC[M_1] \subseteq NC[U]$, the edge unfoldings always overlaps.

#(Non-overlapping edge unfoldings)

Subsetting technique



All edge unfoldings

The family of sets that contains all elements of $NC[M_1]$

The edge unfoldings that do not include M_1 as a partial unfolding

$NC[U_1] = \{e_0, e_1, \dots, e_6, a, \dots, y\}$ $NC[M_1] = \{e_0, e_1, \dots, e_6\}$

Key point:

If $NC[M_1] \subseteq NC[U]$, the edge unfoldings always overlaps.

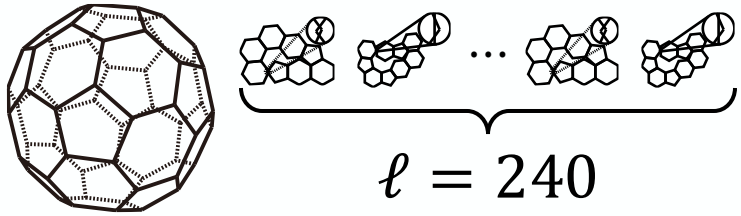
#(Non-overlapping edge unfoldings)

- MOPUs M_k can be enumerated using rotational unfolding.

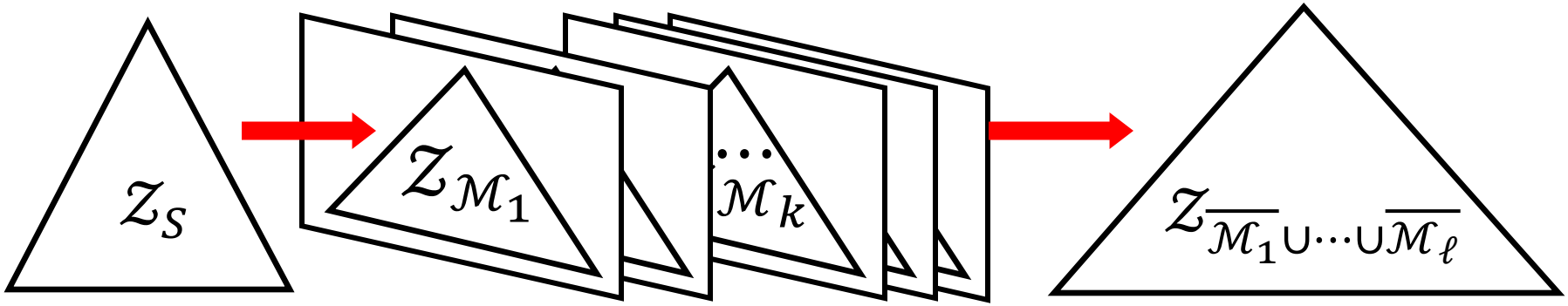
  [T. Shiota and T. Saitoh, 2024]

ℓ: #(MOPUs) in a polyhedron Q

Truncated icosahedron



- Apply the subsetting technique to all MOPUs M_k $(1 \leq k \leq \ell)$

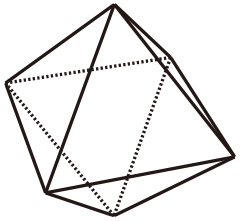


The ZDD representing non-overlapping edge unfoldings.

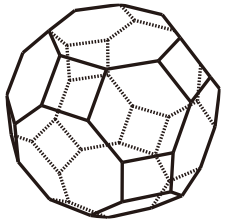
The edge unfoldings that do not include any M_k as a partial unfolding

Conclusion

- Developed an algorithm to count the number of non-overlapping edge unfoldings
- Applied the algorithm to the 175 types of convex regular-faced polyhedra

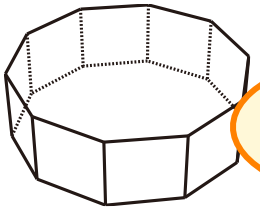


Platonic solids
(5 types / 0 types)



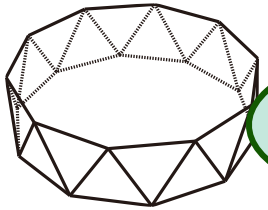
Archimedean solids
(7 types / 6 types)

3 types



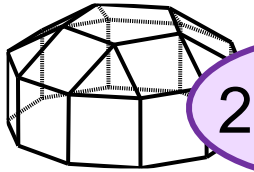
76 types

n-gonal Archimedean prisms
($3 \leq n \leq 23$ / $n \geq 24$)



68 types

m-gonal Archimedean antiprisms
($3 \leq m \leq 11$ / $m \geq 12$)

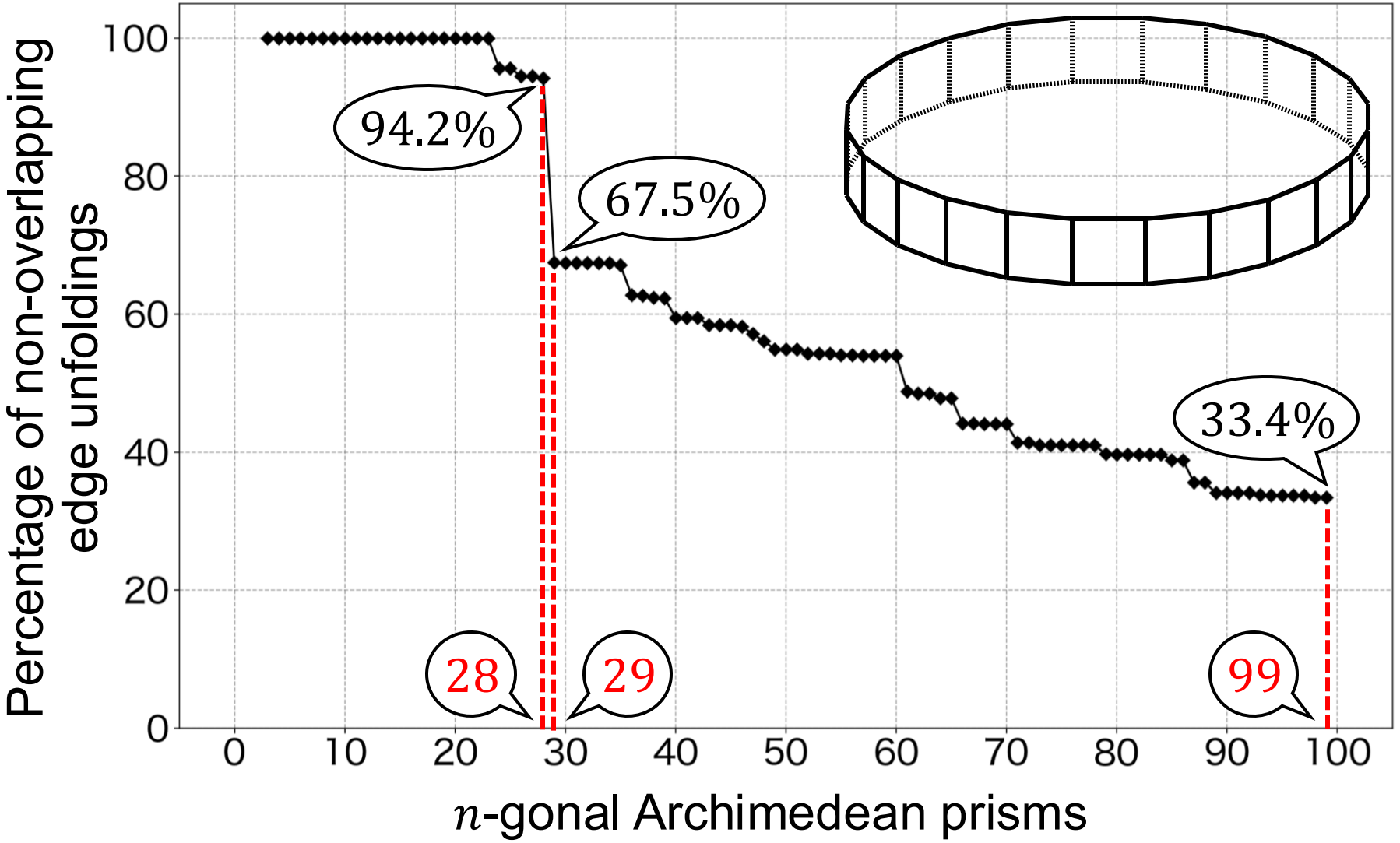


28 types

Johnson solids
(48 types / 44 types)

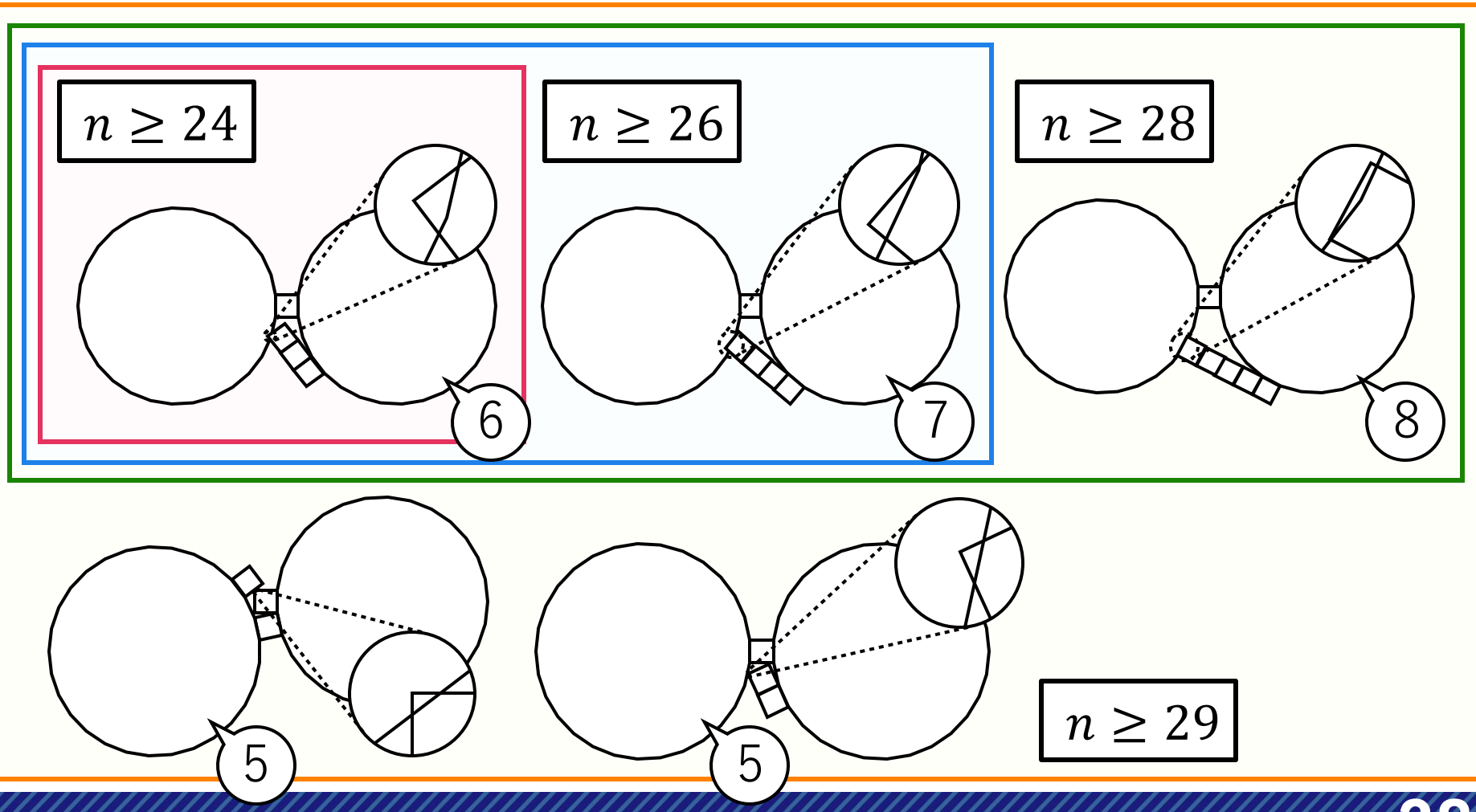
Appendix

Cause of the sharp percentage drop

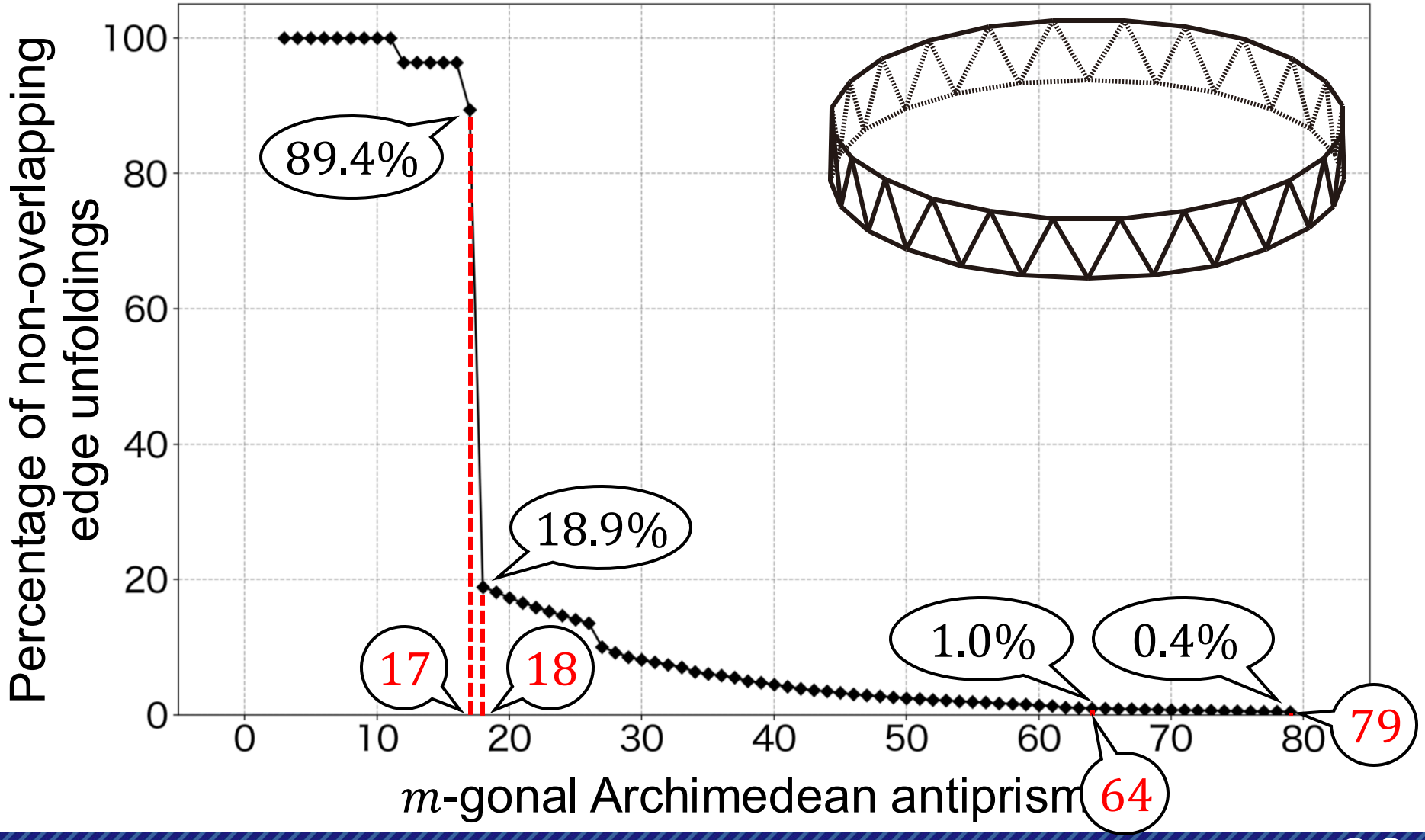


Cause of the sharp percentage drop

Types of MOPUs for Archimedean n-gonal prisms

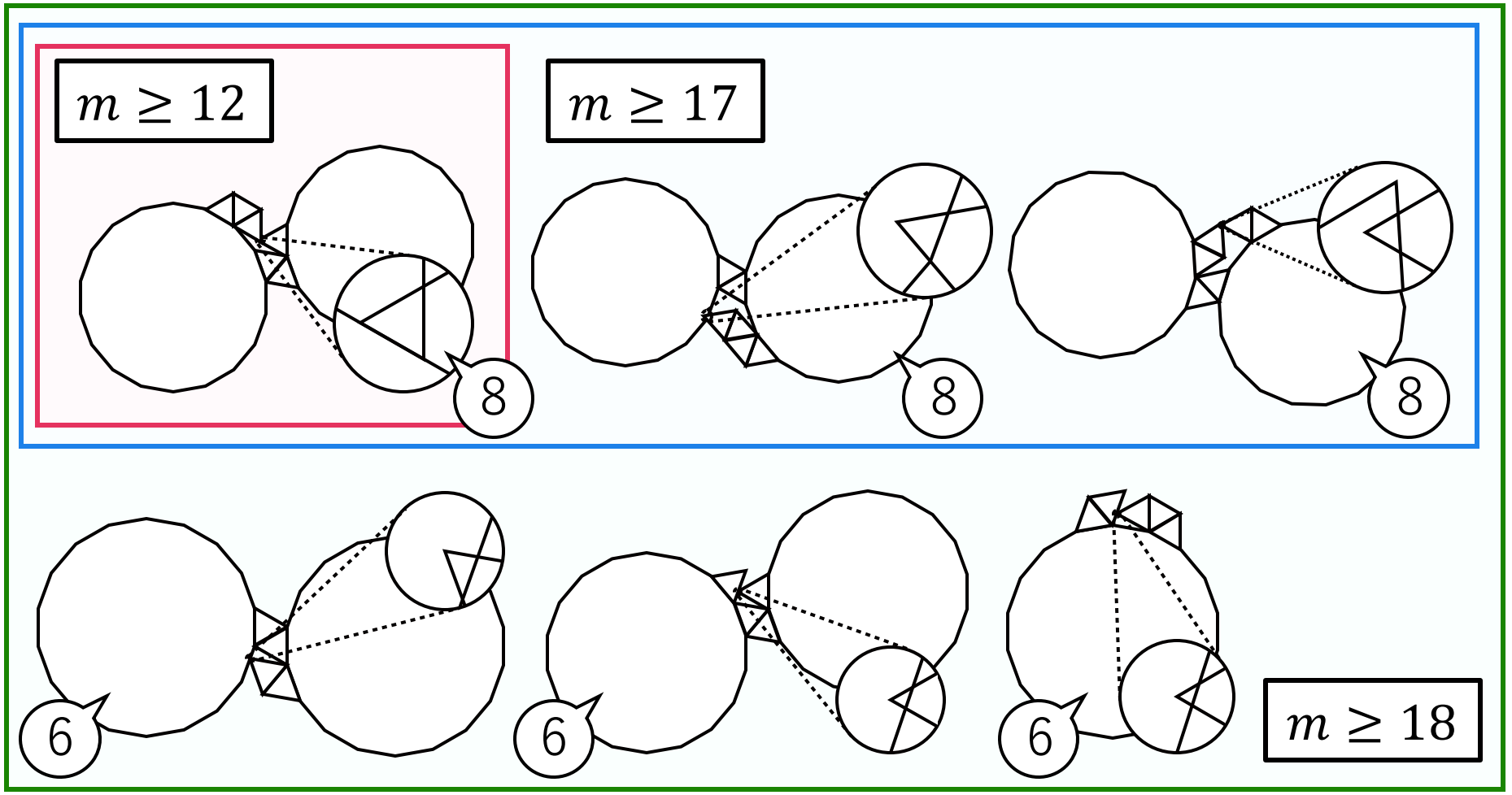


Cause of the sharp percentage drop



Cause of the sharp percentage drop

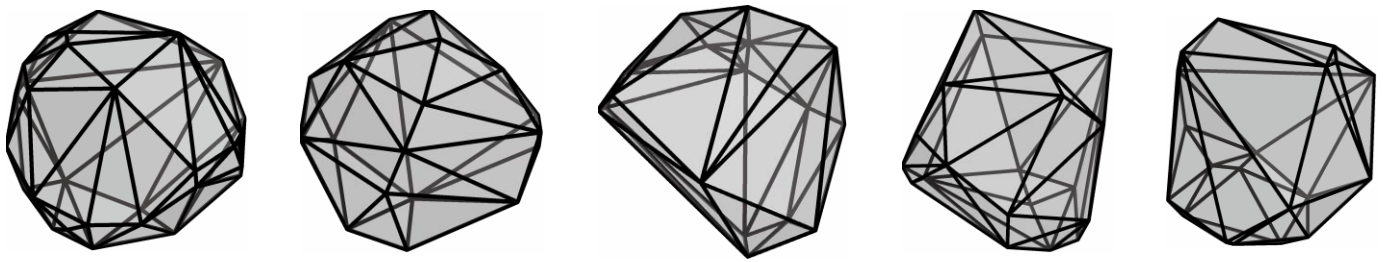
Types of MOPUs for Archimedean m-gonal antiprisms



Background of this study

Open problem (Dürer's problem) [G. C. Shephard, 1975]

Can every convex polyhedron be edge-unfolded without overlaps?



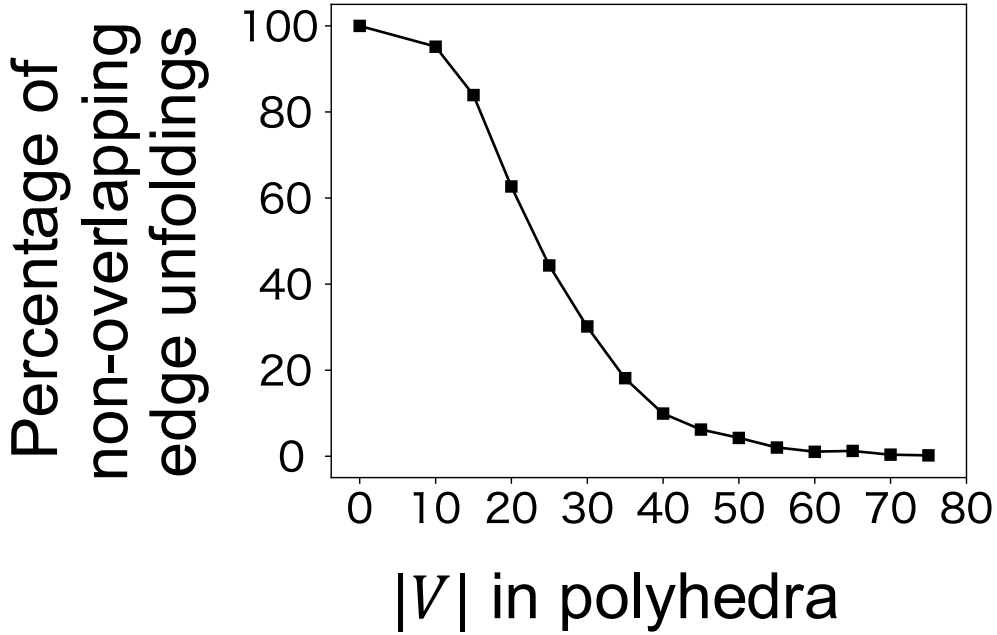
Key points for solve this problem

- To find a convex polyhedron that overlaps in any possible edge unfolding.
- To present an edge-unfolding algorithm that applies to all convex polyhedra.

Background of this study

[C. A. Schevon, 1989]

For randomly generated convex polyhedra, the percentage of non-overlapping edge unfoldings decreases as the number of vertices increases.



Each point …

- Shows the average percentage of non-overlapping edge unfoldings.
- Uses data from 1,000 random unfoldings for each of five polyhedra.